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Staggered Runge-Kutta Schemes for Semilinear Wave Equations

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Abstract

A staggered Runge-Kutta (staggered RK) scheme is the time integration Runge-Kutta type scheme based on staggered grid, which was proposed by Ghrist and Fornberg and Driscoll in 2000. Afterwords, Vewer presented efficiency of the scheme for linear and semilinear wave equations through numerical experiments. We study stability and convergence properties of this scheme for semilinear wave equations. In particular, we prove convergence of a fully discrete scheme obtained by applying the staggered RK scheme to the MOL approximation of the equation.

Key words: Wave equations, Explicit time integration, Staggered Runge-Kutta schemes, Convergence, Stability Analysis

1. Introduction

We consider initial-boundary value problems of the form

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= D \Delta u + g(t, x, u), \quad 0 \leq t \leq T, \quad x \in \Omega, \\
\Phi_y u &= \varphi(t, x), \quad 0 \leq t \leq T, \quad x \in \partial\Omega, \\
u(0, x) &= u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x), \quad x \in \Omega.
\end{align*}
\]

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Here \( u(t, x) \) is an \( \mathbb{R} \)-valued unknown function, \( \Omega \) is a bounded domain in \( \mathbb{R}^i \), \( i = 1, 2, 3 \) with the boundary \( \partial \Omega \), \( \Delta \) is the Laplace operator, \( D \) is a positive constant, and \( g(x, t, u) \) is an \( \mathbb{R} \)-valued given function. Also, \( \Phi_b \) is a boundary operator and \( u_0(x) \), \( v_0(x) \), \( \varphi(t, x) \) are given functions. Many important wave equations, such as the Klein-Gordon equation (see, e.g., [10], [19]) and the nonlinear Klein-Gordon equation (see [17]), are represented in this form. To apply numerical schemes, we may use the form

\[
\frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = D \Delta u + g(t, x, u), \quad 0 \leq t \leq T, \quad x \in \Omega, \\
\Phi_b u = \varphi(t, x), \quad 0 \leq t \leq T, \quad x \in \partial \Omega, \\
u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega.
\]

(1)

A well-known approach in the numerical solution of wave problems in partial differential equations (PDEs) is the method of lines (MOL) (see [12]). In this approach, PDEs are first discretized in space by finite difference or finite element techniques to be converted into a system of ordinary differential equations (ODEs). Let \( \Omega_h \subset \Omega \) be a grid with mesh width \( h > 0 \), and \( V_h \) be the vector space of all functions from \( \Omega_h \) to \( \mathbb{R} \). An MOL approximation of (1) is written in the form

\[
\frac{du_h(t)}{dt} = v_h(t), \quad \frac{dv_h(t)}{dt} = DL_h u_h(t) + \varphi_h(t) + g_h(t, u_h(t)).
\]

(2)

Here \( u_h \), \( v_h \) are approximation functions of \( u \) and \( v \) such that \( u_h(t) \), \( v_h(t) \) \( \in \) \( V_h \) for \( t \in [0, T] \), \( L_h \) is a difference approximation of \( \Delta \), \( g_h \) is a function from \( [0, T] \times V_h \) to \( V_h \) defined by \( g_h(t, u_h)(x) = g(t, x, u_h(t)) \), \( x \in \Omega_h \), for \( t \in [0, T] \), \( u_h \in V_h \), and \( \varphi_h(t) \) is a function determined from the boundary condition.

In order to get the stable numerical solution of (2), Ghrist et al. introduced time-staggered schemes which is based on the idea of the staggered grid. The staggered grid is used to get explicit stable schemes in many fields. For example, the FDTD scheme (see [18]) in the electromagnetic field analysis and the SMAC scheme (see, e.g., [3], [9]) in the fluid calculation use staggered grid in space discretization. To the contrary, Ghrist et al. [5] consider staggered grid in time discretization and introduced the staggered Runge-Kutta (staggered RK) schemes. In particular, they proposed a forth-order, explicit, staggered RK scheme (RKS4) and studied stability and convergence of staggered RK schemes applied to ODEs. Vewer. (see, [15], [16]) presented efficiency of RKS4 for linear and semilinear wave equations through numerical experiments. As is well known, RK approximations for PDEs suffer from
order reduction phenomena. That is, the order of time-stepping in the fully
discrete scheme is, in general, less than that of the underlying RK scheme
(see, e.g., [8], [11], [14] on order reduction phenomena of RK schemes in the
PDE context). Vewer observes the order of RKS4 is three, while that of the
classical RK scheme is two. He also gives an analysis of this phenomenon.
In this paper, we study stability and convergence of staggered RK schemes
for (2). Specifically, we introduce a new stability condition which guaran-
tees the boundedness of numerical solutions and prove convergence of the
schemes.
The paper is organized as follows. In the next section (Section 2), we in-
troduce some notation, including the form of the staggered RK schemes.
In Section 3, we prove a theorem which describes the boundedness of the
numerical solution. In Section 4, we prove a theorem which describes con-
vergence of the scheme applied to (2). In Section 5, we estimate the order of
convergence by using a numerical experiment.

2. Preliminaries

Let \( \tau > 0 \) be a step size. We define the step points \( t_n = n\tau \), \( t_{n+1/2} = (n + 1/2)\tau \) for integer \( n \geq 0 \).
As [5], for positive integer \( s \), a staggered RK scheme for ODEs of the form
\[
\begin{align*}
  u' &= f(t, v) \\
  v' &= g(t, u), \quad 0 \leq t \leq T, \ u, v \in \mathbb{R}
\end{align*}
\]
is given as

\[ v_{n+1/2,1} = v_{n+1/2}, \]

\[ u_{n,i} = u_n + \tau \sum_{j=1}^{i} b_{i,j} f(t_{n+1/2} + e_j \tau, v_{n+1/2,j}), \quad i = 1, \ldots, s - 1, \]

\[ v_{n+1/2,i} = v_{n+1/2} + \tau \sum_{j=1}^{i-1} a_{i,j} g(t_n + c_j \tau, u_{n,j}), \quad i = 2, \ldots, s, \]  \hfill (4)

\[ u_{n+1} = u_n + \tau \sum_{i=1}^{s} d_i f(t_{n+1/2} + e_i \tau, v_{n+1/2,i}), \]

\[ u'_{n+1,1} = u_{n+1}, \]

\[ v'_{n+1/2,i} = v_{n+1/2} + \tau \sum_{j=1}^{i} b'_{i,j} g(t_{n+1} + e'_j \tau, u'_{n+1,j}), \quad i = 1, \ldots, s - 1, \]

\[ u'_{n+1,i} = u_{n+1} + \tau \sum_{j=1}^{i-1} a'_{i,j} f(t_{n+1/2} + e'_j \tau, v'_{n+1/2,j}), \quad i = 2, \ldots, s, \]

\[ v_{n+3/2} = v_{n+1/2} + \tau \sum_{i=1}^{s} d'_i g(t_{n+1} + e'_i \tau, u'_{n+1,i}) \]  \hfill (5)

and the abscissae

\[ c_i = \sum_{j=1}^{i} b_{i,j}, \quad c'_i = \sum_{j=1}^{i} b'_{i,j}, \quad i = 1, \ldots, s - 1, \]

\[ e_i = \sum_{j=1}^{i-1} a_{i,j}, \quad e'_i = \sum_{j=1}^{i-1} a'_{i,j}, \quad i = 2, \ldots, s. \]  \hfill (6)

Here \( a_{i,j}, b_{i,j}, a'_{i,j}, b'_{i,j}, c_i, c'_i, d_i, d'_i, e_i, e'_i \) are coefficients, \( e_1 = e'_1 = 0 \), \( u_{n,i}, v_{n+1/2,i}, u'_{n+1,i}, v'_{n+1/2,i} \) are intermediate variables, \( u_n \) and \( v_{n+1/2} \) are approximate values of \( u(t_n) \) and \( v(t_{n+1/2}) \), respectively.

We describe the algorithm of the staggered RK scheme. In the first step, we calculate \( u_1 \) from \( u_0 \) and \( v_{1/2} \) by (4), where \( v_{1/2} \) is produced by given initial values \( u_0(x) = u_0, \ v_0(x) = v_0, \ x \in \Omega_h \) and using the Runge-Kutta scheme. The next step, we calculate \( v_{3/2} \) from \( v_{1/2} \) and \( u_1 \) by (5). By this way, we calculate \( u_{n+1} \) from \( u_n \) and \( v_{n+1/2} \) by (4), and \( v_{n+3/2} \) from \( v_{n+1/2} \) and \( u_{n+1} \).
Figure 1: The approximation values of staggered RK schemes

by (5).

Fig. 1 shows this process. The solid arrow describes the process of calculating $u_{n+1}$ and the dashed arrow describes the process of calculating $v_{n+3/2}$. All the approximate values are calculated explicitly.

We introduce some notation. The $m \times m$ identity matrix will be denoted by $I_m$. We use the standard symbol $\mathbf{1} = (1, \cdots , 1)^T \in \mathbb{R}^s$.

To estimate stability of the scheme, we use the following linear test equation:

$$
\begin{align*}
\begin{cases}
  u'(t) = v(t) \\
  v'(t) = -\omega^2 u(t)
\end{cases}, \quad \omega \in \mathbb{R}_{\geq 0}
\end{align*}
$$

(7)

with $\mathbb{R}_{\geq 0} = \{ x; x \geq 0, x \in \mathbb{R} \}$.

Applying (4)-(5) to (7), we get

$$
\begin{align*}
  V_{n+1/2} &= v_{n+1/2} - \tau \omega^2 AU_n, \\
  U_n &= u_n + \tau BV_{n+1/2}, \\
  u_{n+1} &= u_n + \tau dV_{n+1/2}, \\
  U'_{n+1} &= u_{n+1} + \tau A'V_{n+1/2}', \\
  V'_{n+1/2} &= v_{n+1/2} - \tau \omega^2 B'U'_{n+1}, \\
  v_{n+3/2} &= v_{n+1/2} - \tau \omega^2 dU'_{n+1},
\end{align*}
$$

(8)
where

\[
A = \begin{pmatrix}
0 & a_{2,1} & 0 & O \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & \ddots \\
& a_{s,1} & \cdots & a_{s,s-1} & 0
\end{pmatrix},
B = \begin{pmatrix}
b_{1,1} & b_{2,1} & 0 & O \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & \ddots \\
b_{s,1} & b_{s,2} & \cdots & b_{s,s}
\end{pmatrix},
\quad d = \begin{pmatrix}
d_1 \\
& \vdots \\
& \vdots \\
d_s
\end{pmatrix}^T
\]

\[
A' = \begin{pmatrix}
0 & a'_{2,1} & 0 & O \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & \ddots \\
& a'_{s,1} & \cdots & a'_{s,s-1} & 0
\end{pmatrix},
B' = \begin{pmatrix}
b'_{1,1} & b'_{2,1} & 0 & O \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & \ddots \\
b'_{s,1} & b'_{s,2} & \cdots & b'_{s,s}
\end{pmatrix},
\quad d' = \begin{pmatrix}
d'_1 \\
& \vdots \\
& \vdots \\
d'_s
\end{pmatrix}^T
\]

\[
V_{n+1/2} = (v_1 + e_1, v_{n+1/2} + e_{n+1/2}, \ldots, v_{n+1/2} + e_{n+1/2})^T,
U_n = (u_1, u_2, \ldots, u_n)^T,
\]

\[
V'_{n+1/2} = (v'_1 + e'_1, v'_{n+1/2} + e'_{n+1/2}, \ldots, v'_{n+1/2} + e'_{n+1/2})^T,
U'_{n+1} = (u'_1 + e'_1, u'_{n+1/2} + e'_{n+1/2}, \ldots, u'_{n+1/2})^T.
\]

Eliminating \( V_{n+1/2}, U_n, U'_{n+1} \) and \( V'_{n+1/2} \), we can rewrite (8) as

\[
\begin{pmatrix}
u_{n+1} \\
v_{n+3/2}
\end{pmatrix} = \begin{pmatrix}
\omega & 0 \\
0 & 1
\end{pmatrix}^{-1} R(\tau \omega) \begin{pmatrix}
\omega & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
u_{n+1} \\
v_{n+1/2}
\end{pmatrix}.
\]

(9)

For \( \theta \geq 0 \), \( R(\theta) \) is given by

\[
R(\theta) = \begin{pmatrix}
1 + r_{1,1}(\theta) & r_{1,2}(\theta) \\
r_{1,2}'(\theta) & 1 + r_{1,2}'(\theta) + 1
\end{pmatrix}
\]

(10)

with

\[
r_{1,1}(\theta) = -\theta^2 d(I_s + \theta^2 AB)^{-1} A,
\quad r_{1,2}(\theta) = \theta d(I_s + \theta^2 AB)^{-1},
\]

\[
r'_{1,1}(\theta) = -\theta^2 d'(I_s + \theta^2 A'B')^{-1} A',
\quad r'_{1,2}(\theta) = -\theta d'(I_s + \theta^2 A'B')^{-1}.
\]

Noticing \((\theta^2 AB)^s = O\) and \((\theta^2 A'B')^s = O\), we get

\[
(I_s + \theta^2 AB)^{-1} = \sum_{i=0}^{s-1} (-\theta^2 AB)^i,
\quad (I_s + \theta^2 A'B')^{-1} = \sum_{i=0}^{s-1} (-\theta^2 A'B')^i
\]
with \((-\theta^2 AB)^0 = (-\theta^2 A'B')^0 = I_s\). Then we rewrite the coefficients in (10) as

\[
\begin{align*}
r_{1,1}(\theta) &= d \sum_{i=0}^{s-1} (-\theta^2)^{i+1} (AB)^i A, \\
r_{1,2}(\theta) &= d \sum_{i=0}^{s-1} (-\theta^2)^i \theta (AB)^i,
\end{align*}
\]

(11)

\[
\begin{align*}
r'_{1,1}(\theta) &= d' \sum_{i=0}^{s-1} (-\theta^2)^{i+1} (A'B')^i A', \\
r'_{1,2}(\theta) &= -d' \sum_{i=0}^{s-1} (-\theta^2)^i \theta (A'B')^i.
\end{align*}
\]

Let \(\lambda_{\pm} = \lambda_{\pm}(\theta)\) be the eigenvalues of (10). We know these eigenvalues are roots of

\[
\lambda^2 - (2 + r_{1,1}(\theta)) \mathbf{1} + r'_{1,1}(\theta) \mathbf{1} + r_{1,2}(\theta) r'_{1,2}(\theta) \mathbf{1} \lambda \\
+ (1 + r_{1,1}(\theta)) (1 + r'_{1,1}(\theta)) \mathbf{1} = 0.
\]

(12)

Under this notation, we define the stability interval of the scheme.

**Definition 1.** The stability interval \(S\) of a staggered RK scheme (4)-(5) is defined by a connected closed interval of \(\{\theta; |\lambda_{\pm}(\theta)| \leq 1, \ \theta \geq 0\}\), which includes 0.

The simplest example of staggered RK schemes is the (staggered) leapfrog scheme (see, e.g., [15])

\[
\begin{align*}
u_{n+1} &= u_n + \tau f(t_{n+1/2}, v_{n+1/2}), \\
v_{n+3/2} &= v_{n+1/2} + \tau g(t_{n+1}, u_{n+1}).
\end{align*}
\]

(13)

This scheme is of order 2 for ODEs. In this case, the scheme for (7) is reduced to (9) with

\[
\begin{align*}
r_{1,1}(\theta) \mathbf{1} &= r'_{1,1}(\theta) \mathbf{1} = 0, \\
r_{1,2}(\theta) \mathbf{1} &= \theta, \\
r'_{1,2}(\theta) \mathbf{1} &= -\theta.
\end{align*}
\]

(14)

Substituting (14) into (12), we get \(\lambda^2 - (2 - \theta^2) \lambda + 1 = 0\). Since the discriminant of \(\lambda^2 - (2 - \theta^2) \lambda + 1 = 0\) is \(D(\theta) = (2 - \theta^2)^2 - 4\), it is easy to see that \(|\lambda_{\pm}(\theta)| \leq 1\) iff \(D(\theta) \leq 0\). \(S\) is estimated by using the smallest positive root of \(-2 = 2 - \theta^2\), i.e. \(S = [0, 2]\).

RKS4 is another example of staggered RK schemes (see, [5]). This scheme is given by taking

\[
A = A' = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = B' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d = d' = \begin{pmatrix} 11/12 & 1/24 & 1/24 \end{pmatrix}.
\]

(15)
This scheme is of order 4 for ODEs. In this case, the scheme for (7) is reduced to (9) with
\[
\begin{align*}
    r_{1,1}(\theta)1 &= r'_{1,1}(\theta)1 = 0, \\
    r_{1,2}(\theta)1 &= \theta - \frac{\theta^3}{24}, \\
    r'_{1,2}(\theta)1 &= -\theta + \frac{\theta^3}{24}.
\end{align*}
\] (16)
Substituting (16) into (12), we get
\[
\lambda^2 - \left\{2 - (\theta - \theta^3/24)^2\right\} \lambda + 1 = 0.
\]
In [15], \( S \) is estimated by using the smallest positive root of \(-2 = 2 - (\theta - \theta^3/24)^2\), i.e. \( S = [0, 2(2^{1/3} + 2^{2/3})] \).

3. Stability of staggered RK schemes

We use (9) to estimate the stability of the staggered RK scheme. In order to prove convergence of the staggered RK scheme in the next section, we have to evaluate \( \|R(\theta)^n\|_2 \) of (10), where \( \| \cdot \|_2 \) is the Euclidean norm on \( \mathbb{R}^2 \) and the corresponding operator norm for \( 2 \times 2 \) matrices. To accomplish this evaluation, we define another stability interval. Let \( \gamma_0 > 0 \) \((\gamma_0 \in S)\) be the smallest positive root of
\[
D(\theta) = r_{1,2}(\theta)1 r'_{1,2}(\theta)1\{r_{1,2}(\theta)1 r'_{1,2}(\theta)1 + 4\} = 0.
\] (17)
By using this \( \gamma_0 \), we define another stability interval \( S' = [0, \gamma_0) \). It is easy to see that \( S' \) is a subset in \( S \). We prove the boundedness of \( \|R(\theta)^n\|_2 \) by using following hypotheses for the staggered RK scheme (4)-(5):

(H1) For \( \theta \in S' \), \( 0 \leq -r'_{1,2}(\theta)1 \leq r_{1,2}(\theta)1 \leq -\gamma_0 r'_{1,2}(\theta)1 \).

(H2) For \( \theta \in S' \), \( D(\theta) \leq 0 \).

(H3) The polynomials \( r_{1,1}(\theta)1 \) and \( r'_{1,1}(\theta)1 \) are 0.

(H4) The following order condition holds: \( d1 = d'1 = 1 \).

The leapfrog scheme (13) and RKS4 (15) satisfy these hypotheses. Substituting (14) into (17), we can take \( \gamma_0 = 2 \) and \( S' = [0, 2) \) for the leapfrog scheme. By (14), the leapfrog scheme satisfies (H1)-(H3). (H4) is checked by using (13). Similarly, we can take \( \gamma_0 = 2\sqrt{6} \) and \( S' = [0, 2\sqrt{6}) \) for RKS4, by substituting (14) and (16) into (17). By (16), RKS4 satisfies (H1)-(H3). By (15), (H4) holds.
Theorem 3.1. Let $\gamma > 0$ be $\gamma < \gamma_0$. Assume that the coefficients $a_{i,j}$, $a'_{i,j}$, $b_{i,j}$, $b'_{i,j}$, $c_i$, $c_i'$, $d_i$, $d_i'$, $e_i$, $e_i'$ in (4)-(5) satisfy (H1)-(H4). Then, there is a positive constant $C$ such that

$$||R(\theta)^n||_2 \leq C$$

holds for any $0 \leq \theta \leq \gamma$ and $n \in \mathbb{N}$. Here $R(\theta)$ is the matrix of (10).

**Proof.** By (H3), we can rewrite

$$R(\theta) = \begin{pmatrix} 1 & r_{1,2}(\theta)1 \\ r'_{1,2}(\theta)1 & 1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1 \end{pmatrix}.$$  

(19)

If $\theta = 0$, $R(\theta)$ is the identity matrix. Then (18) holds for $C = 1$. Let $\theta > 0$. We can diagonalize (19) as

$$R(\theta) = Q(\theta) \begin{pmatrix} \lambda_+(\theta) & 0 \\ 0 & \lambda_-(\theta) \end{pmatrix} Q(\theta)^{-1}.$$  

(20)

Here

$$\lambda_\pm(\theta) = \lambda_\pm = \frac{2 + r_{1,2}(\theta)1r'_{1,2}(\theta)1 \pm \sqrt{D(\theta)}}{2},$$

(21)

$$Q(\theta) = \frac{1}{r'_{1,2}(\theta)1} \begin{pmatrix} \lambda_+ - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) & \lambda_- - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) \\ r'_{1,2}(\theta)1 & r'_{1,2}(\theta)1 \end{pmatrix},$$

$$Q(\theta)^{-1} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} r'_{1,2}(\theta)1 & -\lambda_- + (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) \\ -r'_{1,2}(\theta)1 & \lambda_+ - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) \end{pmatrix}.$$  

Since $\theta \in S$, we have $|\lambda_\pm| \leq 1$. By (H2), the adjoint matrices of $Q(\theta)$ and $Q(\theta)^{-1}$ are

$$Q(\theta)^* = \frac{1}{r'_{1,2}(\theta)1} \begin{pmatrix} \lambda_- - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) & \lambda_+ - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1) \\ r_{1,2}(\theta)1 & r_{1,2}(\theta)1 \end{pmatrix},$$

$$Q(\theta)^{-1*} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} r_{1,2}(\theta)1 & -\lambda_+ + (1 + r_{1,2}(\theta)1r_{1,2}(\theta)1) \\ -r_{1,2}(\theta)1 & \lambda_- - (1 + r_{1,2}(\theta)1r_{1,2}(\theta)1) \end{pmatrix}.$$  

Putting

$$a(\theta) = \{\lambda_- - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1)\}\{\lambda_+ - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1)\},$$

$$b_\pm(\theta) = \{\lambda_\pm - (1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1)\}^2 + (r'_{1,2}(\theta)1)^2,$$

$$c(\theta) = -r'_{1,2}(\theta)1\{\lambda_+ + \lambda_- - 2(1 + r_{1,2}(\theta)1r'_{1,2}(\theta)1)\},$$

9
we have

\[
Q(\theta)^*Q(\theta) = \frac{1}{(r_{1,2}(\theta)\mathbf{1})^2} \left( \begin{array}{cc} a(\theta) + (r'_{1,2}(\theta)\mathbf{1})^2 & b_-(\theta) \\ b_+(\theta) & a(\theta) + (r'_{1,2}(\theta)\mathbf{1})^2 \end{array} \right),
\]

\[
(Q(\theta)^{-1})^*(Q(\theta)^{-1}) = \frac{-1}{\{\lambda_- - \lambda_+\}^2} \left( \begin{array}{cc} 2r'_{1,2}(\theta)^2 & c(\theta) \\ c(\theta) & 2a(\theta) \end{array} \right).
\]

Then, the eigenvalues of \(Q(\theta)^*Q(\theta)\) and \((Q(\theta)^{-1})^*(Q(\theta)^{-1})\) are

\[
a(\theta) + (r'_{1,2}(\theta)\mathbf{1})^2 \pm \sqrt{b_-(\theta)b_+(\theta)}
\]

\[
r_{1,2}(\theta)\mathbf{1},
\]

\[
a(\theta) + (r'_{1,2}(\theta)\mathbf{1})^2 \pm \sqrt{\frac{(a(\theta) + (r'_{1,2}(\theta)\mathbf{1})^2)^2 - 4a(\theta)(r'_{1,2}(\theta)\mathbf{1})^2 + c(\theta)^2}{-\{\lambda_- - \lambda_+\}^2}}
\]

respectively. Putting

\[
\alpha(\theta) = -r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} + (r'_{1,2}(\theta)\mathbf{1})^2,
\]

\[
\beta(\theta) = r'_{1,2}(\theta)\mathbf{1}(\lambda_+ - \lambda_-)i,
\]

these eigenvalues are rewritten as

\[
\frac{\alpha(\theta) \pm \sqrt{\alpha(\theta)^2 - \beta(\theta)^2}}{(r_{2,1}(\theta)\mathbf{1})^2},
\]

\[
\frac{(r_{2,1}(\theta)\mathbf{1})^2 \left\{ \alpha(\theta) \pm \sqrt{\alpha(\theta)^2 - \beta(\theta)^2} \right\}}{\beta(\theta)^2},
\]

respectively. Then, by (20), we have

\[
||R(\theta)^n||_2 \leq ||Q(\theta)||_2 \left| |Q(\theta)^{-1}|_2 \right| = \left| \frac{\alpha(\theta) + \sqrt{\alpha(\theta)^2 - \beta(\theta)^2}}{\beta(\theta)} \right| \leq 2 \left| \frac{\alpha(\theta)}{\beta(\theta)} \right| + 1.
\]

(23)

Substituting (21) into (22) and using (H1), we have

\[
\frac{\left| \alpha(\theta) \right|}{\beta(\theta)} = \frac{|r_{1,2}(\theta)\mathbf{1} - r'_{1,2}(\theta)\mathbf{1}|}{\sqrt{-r_{1,2}(\theta)\mathbf{1}r_{1,2}(\theta)\mathbf{1}(r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} + 4)}} \leq \frac{(1 + \gamma_0)r'_{1,2}(\theta)\mathbf{1}}{r'_{1,2}(\theta)\mathbf{1}\sqrt{r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} + 4}}
\]

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for any $\theta \in [0, \gamma_0]$. By (H1) and (H2), we get $-4 \leq r_{1,2}(\theta)1r_{1,2}'(\theta)1 \leq 0$. As $r_{1,2}(\theta)1r_{1,2}'(\theta)1$ is a polynomial of $\theta$, there exits a minimum value of $r_{1,2}(\theta)1r_{1,2}'(\theta)1 + 4$ in $[0, \gamma_0]$. Let $\gamma_1$ be the value of $\theta$ that gives the minimum value of $r_{1,2}(\theta)1r_{1,2}'(\theta)1 + 4$. We get

$$\left| \frac{\alpha(\theta)}{\beta(\theta)} \right| \leq \frac{1 + \gamma_0}{\sqrt{r_{1,2}(\gamma_1)1r_{2,1}(\gamma_1)1 + 4}}.$$ 

Then, this, together with (23), gives (18) with $C = \frac{2(1 + \gamma_0)}{\sqrt{r_{1,2}(\gamma_1)1r_{2,1}(\gamma_1)1 + 4}} + 1$.

4. Convergence of fully discrete schemes

We assume the following hypotheses for $L_h$:

$L_h$ is a negative definite symmetric matrix.

There exits $h_0 > 0$ and $C_3 > 0$ such that any eigenvalues of $L_h$ is less than $-C_3$ for any $h < h_0$.

Form these hypotheses, we can take a positive definite symmetric matrix $W_h$ satisfying $-DL_h = W_h^2$. Any eigenvalues of $W_h^{-1}$ is less than $1/\sqrt{DC_3}$ for any $h < h_0$.

Using $W_h$, we can rewrite (2) as

$$\frac{du_h(t)}{dt} = v_h(t), \quad \frac{dv_h(t)}{dt} = -W_h^2 u_h(t) + \varphi_h(t) + g_h(t, u_h(t)). \quad (24)$$

In this paper, $\| \cdot \|_{W_h}$ denotes a discrete energy norm (see, e.g., [1], [2]), given by

$$\|(u_h, v_h)^T\|_{W_h}^2 = \|W_h u_h\|^2 + \|v_h\|^2 \quad \text{for any } u_h, v_h \in V_h, \quad (25)$$

where $\| \cdot \|$ denotes the discrete version of the $L_2$-norm in $V_h$, given by

$$\|u_h\|^2 = h \sum_{x \in \Omega_h} \{(u_h)_x\}^2$$

and the corresponding operator norm for $m \times m$ matrices with $m = \dim V_h$.

We define the spatial truncation error $\alpha_h(t)$ by

$$\alpha_h(t) = v'_h(t) + W_h^2 u_h(t) - \varphi_h(t) - g_h(t, u_h(t)), \quad (26)$$
where $u_h(t)$, $v_h(t)$ are $V_h$-valued functions obtained by
restricting the variable $x$ of the exact solutions $u$, $v$ onto $\Omega_h$.
By applying (4)-(5) to (24), we obtain the following scheme for
the problem (1):

$$
\begin{align*}
V_{n+1/2} &= 1'v_{n+1/2} + \tau A\{-W_h^2U_n + \varphi_h(t_n) + g_n\}, \\
U_n &= 1'u_n + \tau BV_{n+1/2}, \\
u_{n+1} &= u_n + \tau dV_{n+1/2}, \\
U'_{n+1} &= 1'u_{n+1} + \tau A'V'_{n+1/2}, \\
V'_{n+1/2} &= 1'v_{n+1/2} + \tau B'\{-W_h^2U'_{n+1} + \varphi_h(t_{n+1}) + g_{n+1}\}, \\
v_{n+3/2} &= v_{n+1/2} + \tau d'\{-W_h^2U'_{n+1} + \varphi_h(t_{n+1}) + g_{n+1}\}.
\end{align*}
$$

Here $1'$ denotes $1 \otimes I_m$ for $1 = (1, \ldots, 1)^T \in \mathbb{R}^s$,

$$
\begin{align*}
A &= A \otimes I_m, & B &= B \otimes I_m, & d &= d \otimes I_m, & A' &= A' \otimes I_m, & B' &= B' \otimes I_m, \\
V_{n+1/2} &= (v_{n+1/2,1}, v_{n+1/2,2}, \ldots, v_{n+1/2,m})^T, & U_n &= (u_{n,1}, u_{n,2}, \ldots, u_{n,m})^T, \\
V'_{n+1/2} &= (v'_{n+1/2,1}, v'_{n+1/2,2}, \ldots, v'_{n+1/2,m})^T, & U'_{n+1} &= (u'_{n+1,1}, u'_{n+1,2}, \ldots, u'_{n+1,m})^T, \\
\varphi_h(t_n) &= (\varphi_h(t_{n,1})^T, \varphi_h(t_{n,2})^T, \ldots, \varphi_h(t_{n,m})^T)^T, & d' &= d' \otimes I_m, \\
g_n &= (g_h(t_{n,1}, u_{n,1})^T, g_h(t_{n,2}, u_{n,2})^T, \ldots, g_h(t_{n,m}, u_{n,m})^T)^T, & W_h &= I_s \otimes W_h
\end{align*}
$$

with $\otimes$ standing for the Kronecker product (see, e.g., [4]), $u_{n,i}$, $v_{n+1/2,i}$, $u'_{n+1,i}$
and $v'_{n+1/2,i}$ are intermediate variables, $t_{n,j} := t_n + c_j \tau$, $t_{n+1,j} := t_{n+1} + c_j \tau$,
$u_n$ and $v_{n+1/2}$ are approximate values of $u_h(t_n)$ and $v_h(t_{n+1/2})$, respectively.
For some $s$-dimensional vector $a = (a_1, \ldots, a_s)^T$, we define $a' = (a'_1, \ldots, a'_s)^T$.
In addition to the (H1)-(H4), we assume the following hypothesis for
the staggered RK scheme (4)-(5):

(H5) The following order conditions hold:

$$
\begin{align*}
(A1)^2 + A1 &= 2ABA1, & (B1)^2 - B1 &= 2BA1, \\
(A'1)^2 + A'1 &= 2A'B'1, & (B'1)^2 - B'1 &= 2B'A'1, \\
dA1 &= d'A'1 = 0.
\end{align*}
$$

The leapfrog scheme and RKS4 satisfy (H5), which is checked by (13) and (15).
We assume the following condition which gives the restriction for $\tau$ and $h$. 

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(H6) $\tau \rho(W_h) \in S'$. Here $\rho(W_h)$ is a spectral radius of $W_h$.

We put the coefficients of (4)-(5) as

$$
\zeta = \frac{4(A1)^3 + 6(A1)^2 + 3(A1)}{24} - \frac{A(B1)^2}{2},
$$

$$
\eta = \frac{4(B1)^3 - 6(B1)^2 + 3(B1)}{24} - \frac{B(A1)^2}{2},
$$

$$
\zeta' = \frac{4(A'1)^3 + 6(A'1)^2 + 3(A'1)}{24} - \frac{A'(B'1)^2}{2},
$$

$$
\eta' = \frac{4(B'1)^3 - 6(B'1)^2 + 3(B'1)}{24} - \frac{B'(A'1)^2}{2}.
$$

Moreover, we assume the following condition for the problem (1):
The exact solution $u(t, x)$ is of class $C^4$ with respect to $t$, $g(t, x, u)$ is of class $C^3$ with respect to $t$, $u$ and (each component of) the derivative $\partial g/\partial u$ is bounded for $(t, x, u) \in [0, T] \times \Omega \times \mathbb{R}$. For simplicity, we consider a step size of the form $\tau = T/N$ with positive integer $N$. Then, we have the following theorem.

**Theorem 4.1.** Assume that the coefficients $a_{i,j}, a'_{i,j}, b_{i,j}, b'_{i,j}, c_i, c'_i, d_i, d'_i, e_i, e'_i$ in (4)-(5) satisfy $(H1)-(H5)$ and $\tau$ satisfies $(H6)$. Then, there is a positive constant $C_1$ such that

$$
\left\| \left( u_n - u_h(t_n), v_{n+1/2} - v_h(t_{n+1/2}) \right)^T \right\|_{W_h} \leq C_1 \left( \tau^2 + \max_{0 \leq t \leq T} ||\alpha_h(t)|| \right) \quad (28)
$$

holds.

**Proof.** Put

$$
V_h(t_{n+1/2}) = \left( v_h(t_{n+1/2, 1})^T, v_h(t_{n+1/2, 2})^T, \cdots, v_h(t_{n+1/2, s})^T \right)^T,
$$

$$
U_h(t_n) = \left( u_h(t_{n, 1})^T, u_h(t_{n, 2})^T, \cdots, u_h(t_{n, s})^T \right)^T,
$$

$$
V_h(t'_{n+1/2}) = \left( v_h(t'_{n+1/2, 1})^T, v_h(t'_{n+1/2, 2})^T, \cdots, v_h(t'_{n+1/2, s})^T \right)^T,
$$

$$
g_h(t_n) = \left( g_h(t_{n, 1}, u_h)^T, g_h(t_{n, 2}, u_h)^T, \cdots, g_h(t_{n, s}, u_h)^T \right)^T,
$$

where $t_{n+1/2,j} := t_{n+1/2} + e_j \tau$, $t'_{n+1/2,j} := t_{n+1/2} + e'_j \tau$, $j = 1, \cdots, s$. Replacing $U_n, U'_{n+1}, V_{n+1/2}, V'_{n+1/2}, u_n$ and $v_{n+1/2}$ in the scheme (27) with
\[ U_h(t_n), U_h(t_{n+1}), V_h(t_{n+1/2}), V_h(t'_{n+1/2}), u_h(t_n) \text{ and } v_h(t_{n+1/2}), \text{ we obtain the recurrence relation} \]

\[
\begin{align*}
V_h(t_{n+1/2}) &= 1'v_h(t_{n+1/2}) + \tau A\{-W_h^2U_h(t_n) + \varphi_h(t_n) + g_h(t_n)\} + r_{n+1/2}, \\
U_h(t_n) &= 1'u_h(t_n) + \tau BV_h(t_{n+1/2}) + r_n, \\
u_h(t_{n+1}) &= u_h(t_n) + \tau dV_h(t_{n+1/2}) + \rho_n, \\
U_h(t_{n+1}) &= 1'u_h(t_{n+1}) + \tau A'V_h(t'_{n+1/2}) + r_{n+1}, \\
V_h(t'_{n+1/2}) &= 1'v_h(t_{n+1/2}) + \tau B'\{-W_h^2U_h(t_{n+1}) + \varphi_h(t_{n+1}) + g_h(t_{n+1})\} + r'_{n+1/2}, \\
v_h(t_{n+3/2}) &= v_h(t_{n+1/2}) + \tau d'\{-W_h^2U_h(t_{n+1}) + \varphi_h(t_{n+1}) + g_h(t_{n+1})\} + \rho_{n+1/2} \\
\end{align*}
\]

(29)

with the residuals

\[
\begin{align*}
r_n &= (r_{n,1}^T, r_{n,2}^T, \ldots, r_{n,s}^T)^T, \\
r'_{n+1/2} &= (r'_{n+1/2,1}^T, r'_{n+1/2,2}^T, \ldots, r'_{n+1/2,s}^T)^T, \\
\rho_n \text{ and } \rho_{n+1/2}. \text{ By (6), (26), (H4) and (H5), these residuals are expanded as} \]

\[
\begin{align*}
r_{n+1/2} &= \tau^3\zeta v_h^{(3)}(t_{n+1/2}) + \tau A\alpha_h(t_n) + O(\tau^4), \\
r_n &= \tau^3\eta u_h^{(3)}(t_n) + O(\tau^4), \\
\rho_n &= \frac{\tau^3}{2} \left( \frac{1}{12} - d(A'1)^2 \right) u_h^{(3)}(t_n) + O(\tau^4), \\
r'_{n+1} &= \tau^3\zeta' u_h^{(3)}(t_{n+1}) + O(\tau^4), \\
r'_{n+1/2} &= \tau^3\eta' v_h^{(3)}(t_{n+1/2}) + \tau B'\alpha_h(t_{n+1}) + O(\tau^4), \\
\rho'_{n+1/2} &= \frac{\tau^3}{2} \left( \frac{1}{12} - d'(A'1)^2 \right) v_h^{(3)}(t_{n+1/2}) + \tau d'\alpha_h(t_{n+1}) + O(\tau^4). \\
\end{align*}
\]

(30)

Here

\[ \alpha_h(t_n) = (\alpha_h(t_{n,1})^T, \alpha_h(t_{n,2})^T, \ldots, \alpha_h(t_{n,s})^T)^T, \]

\[ \zeta = \zeta \otimes I_m, \quad \eta = \eta \otimes I_m, \quad \zeta' = \zeta' \otimes I_m, \quad \eta' = \eta' \otimes I_m, \]

\]
\(O(\tau^4)\) denotes a term whose component for each \(x \in \Omega_h\) is of \(O(\tau^4)\). Subtracting (27) from (29), we obtain

\[
\begin{align*}
\delta_{n+1/2} &= 1' \varepsilon_{n+1/2} - \tau A(W_h^2 J_n - g_h(t_n) + g_n) + r_{n+1/2}, \\
\delta_n &= 1' \varepsilon_n + \tau B \delta_{n+1/2} + r_n, \\
\varepsilon_{n+1} &= \varepsilon_n + \tau d \delta_{n+1/2} + \rho_n, \\
\delta'_{n+1} &= 1' \varepsilon_{n+1} + \tau A' \delta'_{n+1/2} + r_{n+1}, \\
\delta'_{n+1/2} &= 1' \varepsilon_{n+1/2} - \tau B'(W_h^2 J_{n+1} - g_h(t_{n+1}) + g_{n+1}) + r'_{n+1/2}, \\
\varepsilon_{n+3/2} &= \varepsilon_{n+1/2} - \tau d'(W_h^2 J_{n+1} - g_h(t_{n+1}) + g_{n+1}) + \rho_{n+1/2}.
\end{align*}
\]

Here

\[
\begin{align*}
\delta_{n+1/2} &= V_h(t_{n+1/2}) - V_{n+1/2}, \\
\delta_n &= U_h(t_n) - U_n, \\
\delta'_{n+1} &= U_h(t_{n+1}) - U'_{n+1}, \\
\delta'_{n+1/2} &= V_h(t_{n+1/2}) - V'_{n+1/2}
\end{align*}
\]

for the errors

\[
\begin{align*}
\varepsilon_n &= u_h(t_n) - u_n, \\
\varepsilon_{n+1/2} &= v_h(t_{n+1/2}) - v_{n+1/2}.
\end{align*}
\]

Let \(J_n\) be \(J_n = \text{diag}(J_{n,1}, J_{n,2}, \cdots, J_{n,s})\) and \(J_{n,i}\) be a function from \(\Omega_h\) to \(\mathbb{R}\) whose value for \(x \in \Omega_h\) is

\[
J_{n,i}(x) = \int_0^1 \frac{\partial g}{\partial u}(t_{n,i}, x, (1 - \theta)u_{n,i}(x) + \theta u_h(t_{n,i}, x)) d\theta.
\]

By the assumption that \(\partial g/\partial u\) is bounded, there is a constant \(\gamma_3\) such that

\[
||J_{n,i}v|| \leq \gamma_3 ||v|| \quad \text{for any } v \in V_h,
\]

where the multiplication \(J_{n,i}v\) is component-wise for \(x \in \Omega_h\). Then we obtain

\[
\begin{align*}
\delta_{n+1/2} &= 1' \varepsilon_{n+1/2} - \tau A(W_h^2 J_n - J_n) \delta_n + r_{n+1/2}, \\
\delta_n &= 1' \varepsilon_n + \tau B \delta_{n+1/2} + r_n, \\
\varepsilon_{n+1} &= \varepsilon_n + \tau d \delta_{n+1/2} + \rho_n, \\
\delta'_{n+1} &= 1' \varepsilon_{n+1} + \tau A' \delta'_{n+1/2} + r_{n+1}, \\
\delta'_{n+1/2} &= 1' \varepsilon_{n+1/2} - \tau B'(W_h^2 J_{n+1} - J_{n+1}) \delta'_{n+1} + r'_{n+1/2}, \\
\varepsilon_{n+3/2} &= \varepsilon_{n+1/2} - \tau d'(W_h^2 J_{n+1} - J_{n+1}) \delta'_{n+1} + \rho_{n+1/2}.
\end{align*}
\]
Eliminating $\delta_n$, $\delta_{n+1/2}$, $\delta'_{n+1/2}$ and $\delta_{n+1}$, we have
\[
\begin{pmatrix}
W_h \xi_{n+1} \\
\xi_{n+3/2}
\end{pmatrix} = R_n \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} + M_n \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix}. \tag{32}
\]
Here
\[
R_n = \begin{pmatrix}
I_m + R_{1,1} V' & R_{1,2} V' \\
R_{1,2} V'R_{1,1} V' + R_{1,2} V' & I_m + R_{1,2} V'R_{1,2} V' + R_{1,2} V'
\end{pmatrix}, \quad
M_n = \begin{pmatrix}
I_m & O \\
R_{1,2} V' & I_m
\end{pmatrix}.
\]
\[
R_{1,1} = -\tau d(I + \tau^2 A(W_h^2 - J_n) B)^{-1} A(W_h^2 - J_n),
\]
\[
R_{1,2} = \tau d(I + \tau^2 A(W_h^2 - J_n) B)^{-1} W_h,
\]
\[
R'_{1,1} = -\tau d'(W_h^2 - J_{n+1})(I + \tau^2 A'B'(W_h^2 - J_{n+1}))^{-1} A',
\]
\[
R'_{1,2} = -\tau d'(W_h^2 - J_{n+1})(I + \tau^2 A'B'(W_h^2 - J_{n+1}))^{-1} W_h^{-1},
\]
\[
W_h \xi_n = R_{1,1} W_h r_n + R_{1,2} r_{n+1/2} + W_h \rho_n,
\]
\[
\xi_{n+1/2} = R'_{1,1} W_h r_n + R'_{1,2} r'_{n+1/2} + \rho_{n+1/2}
\]
with $I = I_s \otimes I_m$.

In order to prove the convergence, we introduce new variables following [6] and [15]. As in the proof of Lemma II.2.3 in [6] and 5.3 in [15], we put
\[
\begin{pmatrix}
W_h \nu_n \\
\nu_{n+1/2}
\end{pmatrix} = (R(\tau W_h) - I_{2m})^{-1} M(\tau W_h) \begin{pmatrix}
W_h \psi_n \\
\psi_{n+1/2}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
[d'(I + \tau^2 A'B'W_h^2)^{-1} W_h^{-1} - 1]^{-1} W_h^{-1} \tau^{-1} \psi_{n+1/2} \\
[d(I + \tau^2 A W_h^2 B)^{-1} W_h^{-1} - 1]^{-1} W_h^{-1} \tau^{-1} W_h \psi_{n+1/2}
\end{pmatrix}, \tag{34}
\]
\[
\begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} = \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} + \begin{pmatrix}
W_h \nu_n \\
\nu_{n+1/2}
\end{pmatrix},
\]
\[
\begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} = \tau M(\tau W_h) \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} - \tau \bar{R}_n \begin{pmatrix}
W_h \nu_n \\
\nu_{n+1/2}
\end{pmatrix} + \begin{pmatrix}
W_h(\nu_{n+1} - \nu_n) \\
(\nu_{n+3/2} - \nu_{n+1/2})
\end{pmatrix} \tag{35}
\]
and rewrite (32) as
\[
\begin{pmatrix}
W_h \xi_{n+1} \\
\xi_{n+3/2}
\end{pmatrix} = R_n \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix} + \begin{pmatrix}
W_h \xi_n \\
\xi_{n+1/2}
\end{pmatrix}. \tag{36}
\]
Here

\[
M(\tau W_h) = \begin{pmatrix}
I_m & O \\
r'_{1,2}(\tau W_h) & I_m
\end{pmatrix},
\]

\[
W_h \psi_n = r_{1,1}(\tau W_h) W_h r_n + c_{1,2}(\tau W_h) r_{n+1} + W_h \rho_n,
\]

\[
\psi_{n+1/2} = r'_{1,2}(\tau W_h) W_h r_{n+1} + r'_{1,1}(\tau W_h) r_{n+1/2} + \rho_{n+1/2},
\]

\[
W_h \bar{\xi}_n = \bar{R}_{1,1} W_h r_n + \bar{R}_{1,2} r_{n+1/2},
\]

\[
\bar{\xi}_{n+1/2} = \bar{R}'_{1,2} W_h r_{n+1} + \bar{R}'_{1,1} r'_{n+1/2}.
\]

\(\bar{R}_n\) is defined as \(\tau \bar{R}_n = R_n - R(\tau W_h)\), given by

\[
\bar{R}_n = \begin{pmatrix}
\bar{R}_{1,1} & \bar{R}_{1,1}' \\
\bar{R}_{1,2}' & \bar{R}_{1,2}
\end{pmatrix} = \begin{pmatrix}
\bar{R}_{1,1} & \bar{R}_{1,1}' \\
\bar{R}_{1,2}' & \bar{R}_{1,2}
\end{pmatrix}.
\]

Since \(AW_h^2 B = W_h^2 AB\), \(A'B'W_h^2 = W_h^2 A'B'\), \(\bar{R}_{1,1}\), \(\bar{R}_{1,2}\), \(i = 1, 2\) are written as

\[
\bar{R}_{1,1} = -\tau d \sum_{i=0}^{s-1} (-1)^i \left\{ (\tau^2 W_h^2 AB - \tau^2 AJ_n B)^i - (\tau^2 W_h^2 AB)^i \right\} \{AW_h^2 \}
\]

\[
+ \tau d \sum_{i=0}^{s-1} (\tau^2 A(J_n - W_h^2 B)^i) AJ_n,
\]

\[
\bar{R}_{1,2} = d \sum_{i=0}^{s-1} (-1)^i \left\{ (\tau^2 W_h^2 AB - \tau^2 AJ_n B)^i - (\tau^2 W_h^2 AB)^i \right\} W_h,
\]

\[
\bar{R}'_{1,1} = -\tau d' W_h^2 \sum_{i=0}^{s-1} (-1)^i \left\{ (\tau^2 W_h^2 A'B' - \tau^2 A'B' J_{n+1})^i - (\tau^2 W_h^2 A'B')^i \right\} A'
\]

\[
+ \tau d' J_{n+1} \sum_{i=0}^{s-1} (\tau^2 A'B'(J_{n+1} - W_h^2))^i A',
\]

\[
\bar{R}'_{1,2} = -d' W_h \sum_{i=0}^{s-1} (-1)^i \left\{ (\tau^2 W_h^2 A'B' - \tau^2 A'B' J_{n+1})^i - (\tau^2 W_h^2 A'B')^i \right\}
\]

\[
+ d' J_{n+1} \sum_{i=0}^{s-1} (\tau^2 A'B'(J_{n+1} - W_h^2))^i W_h^{-1}.
\]

By (31) and (H6), we can estimate \(\bar{R}_{1,i}\), \(\bar{R}'_{1,i}\), \(i = 1, 2\) as

\[
\bar{R}_{1,i} = O(\tau), \quad \bar{R}'_{1,1} = O(\tau), \quad \bar{R}'_{1,2} = O(1).
\]

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Substituting (30) into (33) and (38), we get

\[
\left\| (\tilde{\xi}_n, \tilde{\xi}_{n+1/2})^T \right\|_{W_h} \leq C'_1 \left( \tau^2 + \max_{i=0,1} \| \alpha_h(t_{n+i}) \| \right)
\]

with a positive constant \( C'_1 \).

For \( \theta \in S' \), there exit some positive constants \( \gamma_4, \gamma_4' \) such that, \( r_{1,2}(\theta) 1/\theta = d(I_2 + \theta^2 A^2 B)^{-1} 1 > \gamma_4 \) and \( -r'_{1,2}(\theta) 1/\theta = \bar{d}(I_2 + \theta^2 A'B')^{-1} 1 > \gamma_4' \). By (H6), any eigenvalues of \( [d(I + \tau^2 W_h^2 AB)^{-1} 1]^{-1} \) and \( [d(I + \tau^2 W_h^2 AB)^{-1} W_h]^{-1} \) are less than \( \gamma_4 \) and \( \gamma_4' \), respectively. Substituting (30) into (37), \( W_h^{-1} r^{-1} W_h \psi_n \) and \( W_h^{-1} r^{-1} \psi_{n+1/2} \) are represented as

\[
W_h^{-1} r^{-1} W_h \psi_n = r_{1,2}(\tau W_h) A \alpha_h(t_n) + O(\tau^2),
\]

\[
W_h^{-1} r^{-1} \psi_{n+1/2} = (r'_{1,1}(\tau W_h) B' + d') \alpha_h(t_{n+1}) + O(\tau^2).
\]

Substituting (41) into (34), there is a positive constant \( C''_1 \) such that

\[
\left\| (\nu_n, \nu_{n+1/2})^T \right\|_{W_h} \leq C''_1 \left( \tau^2 + \max_{i=0,1} \| \alpha_h(t_{n+i}) \| \right).
\]

Since \( u_h^{(3)}(t_{n+1}) - u_h^{(3)}(t_n) = O(\tau) \) and \( v_h^{(3)}(t_{n+3/2}) - v_h^{(3)}(t_{n+1/2}) = O(\tau) \), we get

\[
W^{-1} r^{-1} W_h(\psi_{n+1} - \psi_n) = \tau r_{1,2}(\tau W_h) A \{ \alpha_h(t_{n+1}) - \alpha_h(t_n) \} + O(\tau^3),
\]

\[
W^{-1} r^{-1} (\psi_{n+3/2} - \psi_{n+1/2}) = \tau (r'_{1,1}(\tau W_h) B' + d') \{ \alpha_h(t_{n+2}) - \alpha_h(t_{n+1}) \} + O(\tau^3).
\]

Thus, by using (35), (40) and (42), there is a positive constant \( C_2 \) such that

\[
\left\| (\tilde{\xi}_n, \tilde{\xi}_{n+1/2})^T \right\|_{W_h} \leq C_2 \left( \tau^3 + \tau \max_{i=0,1} \| \alpha_h(t_{n+i}) \| \right).
\]

Moreover, let \( \omega_j \) be the eigenvalues of \( W_h \). Then, by taking the orthogonal matrix \( P \) to be \( P^{-1}(\tau W_h) P = \text{diag}(\tau \omega_j) \), we have

\[
R(\tau W_h) = P R(\text{diag}(\tau \omega_j)) P^{-1}, \text{ where } P = I_2 \otimes P.
\]

Here \( R(\text{diag}(\tau \omega_j)) \) is the same formula as (10), replacing \( \theta \) by \( \text{diag}(\tau \omega_j) \). Let \( \lambda_{\pm}(\tau \omega_j) = \lambda_{\pm} \) be the eigenvalues of \( R(\text{diag}(\tau \omega_j)) \). \( \lambda_{\pm} \) are the solutions of (12), replacing \( \theta \) by \( \tau \omega_j \). By (H6), we have \( 0 \leq \tau \omega_j < \gamma_0 \) and

\[
|\lambda_{\pm}| \leq 1, \ j = 1, \cdots, m.
\]
Then, by using Theorem 3.1, we obtain
\[ ||R(W_h)\tau^n|| = ||R(diag(\tau\omega_j))\tau^n|| \leq K \] (44)
with \( K \) a constant independent of \( n \in \mathbb{N} \), \( \tau \) and \( h \), \( || \cdot || \) denotes the operator norm for \( 2m \times 2m \) matrices.
By (39), we obtain
\[ ||\bar{R}_n|| \leq K_1, \] (45)
where \( K_1 \) is a constant independent of \( n, \tau \) and \( h \).
From (44) and (45), we obtain
\[ \prod_{i=1}^{n} R_i \leq ||R(W_h)^n||(1 + \tau K_1)^n \leq K e^{\tau \tau K_1} \leq K_2. \] (46)
Hence, from (36), (43) and (46), we obtain
\[ \left( \hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2} \right)^T \leq K_2 \left( \hat{\varepsilon}_0, \hat{\varepsilon}_{1/2} \right)^T + K_2 n C_2 \left( \tau^2 + \tau \max_{0 \leq t \leq T} ||\alpha_h(t)|| \right), \]
which implies that
\[ \left( \hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2} \right)^T \leq K_2 \left( \nu_0, \nu_{1/2} + \nu_{1/2} \right)^T + K_2 T C_2 \left( \tau^2 + \max_{0 \leq t \leq T} ||\alpha_h(t)|| \right) \]
for \( 1 \leq n \leq N \). Using \( \left( \nu_0, \nu_{1/2} + \nu_{1/2} \right)^T \leq C_2' \tau^2 \) for a constant \( C_2' > 0 \),
\[ \left( \varepsilon_n, \varepsilon_{n+1/2} \right)^T \leq \left( \hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2} \right)^T + \left( \nu_n, \nu_{n+1/2} \right)^T \leq \left( \hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2} \right)^T + \left( \nu_n, \nu_{n+1/2} \right)^T \]
and rewriting the constants, we finally obtain (28).

5. Numerical experiments
We examine the convergence of the leapfrog scheme (13) and RKS4 (15), by using the following model problem of the form
\[ \frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(t, x, u), \quad 0 \leq t \leq T, \quad x \in \Omega, \]
\[ u(t, 0) = \beta_0(t), \quad u(t, 1) = \beta_1(t), \quad 0 \leq t \leq T, \]
\[ u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x), \quad x \in \Omega. \] (47)
Here \( T = 1 \), \( \Omega = [0, 1] \), \( g(t, x, u) = -\sin u(t) \) and \( \beta_0(t), \beta_1(t), u_0(x) \) and \( v_0(x) \) are given by using the following exact solution ([13])

\[
    u(t, x) = 4 \tan^{-1} \left\{ \gamma \sinh \left( \frac{x}{\sqrt{1 - \gamma^2}} \right) \big/ \cosh \left( \frac{\gamma t}{\sqrt{1 - \gamma^2}} \right) \right\}
\]

with \( \gamma = 0.5 \). Let \( N \) be a positive integer, \( h = 1/N \), and \( \Omega_h \) be a uniform grid with nodes \( x_j = jh, j = 0, 1, \ldots, N \). We discretize \( \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(t, x, u) \) in space with the fourth-order implicit scheme

\[
    \frac{1}{12} \left\{ \frac{dv^{j-1}(t)}{dt} + 10 \frac{dv^j(t)}{dt} + \frac{dv^{j+1}(t)}{dt} \right\} = \frac{1}{h^2} \left\{ u^{j-1}(t) - 2u^j(t) + u^{j+1}(t) \right\}
\]

\[
    - \frac{1}{12} \left\{ \sin u^{j-1}(t) + 10 \sin u^j(t) + \sin u^{j+1}(t) \right\}
\]

with \( u^j(t) \approx u(t, x_j), v^j(t) \approx v(t, x_j) \) (see, [16]). Putting

\[
    u_h(t) = (u^0(t), \ldots, u^N(t))^T, \quad v_h(t) = (v^0(t), \ldots, v^N(t))^T,
\]

we obtain an MOL approximation

\[
    \frac{du_h(t)}{dt} = v_h(t), \quad \hat{H} \frac{dv_h(t)}{dt} = \hat{L}_h u_h(t) + \hat{\varphi}_h(t) + \hat{H} g_h(t, u_h(t)),
\]

where

\[
    \hat{L}_h = \frac{1}{h^2} \begin{pmatrix}
        -2 & 1 & 0 & \cdots & 0 \\
        1 & -2 & 1 & \cdots & 0 \\
        0 & 1 & -2 & \cdots & 0 \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & 1 & -2
    \end{pmatrix}, \quad
    \hat{H} = \frac{1}{12} \begin{pmatrix}
        10 & 1 & 0 & \cdots & 0 \\
        1 & 10 & 1 & \cdots & 0 \\
        0 & 1 & 10 & \cdots & 0 \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & 1 & 10
    \end{pmatrix},
\]

and \( \hat{\varphi}_h(t) = (\beta_0(t), 0, \ldots, 0, \beta_1(t))^T \). The eigenvalues of \( \hat{L}_h \) and \( \hat{H} \) are

\[
    \frac{2}{h^2} \left( \cos \frac{(j + 1)\pi}{N + 2} - 1 \right), \quad \frac{1}{6} \left( 5 + \cos \frac{(j + 1)\pi}{N + 2} \right), \quad j = 0, 1, \ldots, N,
\]

respectively.

Multiplying \( \hat{H}^{-1} \) to (48), we get (2) with \( D = 1 \), \( L_h = \hat{H}^{-1} \hat{L}_h, \varphi_h(t) = \hat{H}^{-1} \hat{\varphi}_h(t) \). By (49) the eigenvalues of \( L_h \) is

\[
    \frac{12}{h^2} \left( 1 - \frac{6}{5 + \cos((j + 1)\pi/(N + 2))} \right), \quad j = 0, 1, \ldots, N.
\]
Since
\[
\tau \rho(W_h) = \frac{2\sqrt{3}\tau}{h} \left( \frac{6}{5 + \cos((N + 1)\pi/(N + 2))} - 1 \right)^{\frac{1}{2}} < \frac{\sqrt{6}\tau}{h},
\]
if we take the step size \( \tau < \sqrt{2}h/\sqrt{3} \), (H6) holds for the leapfrog scheme. If we take the step size \( \tau < 2h \), (H6) holds for RKS4. We take the various grid and step size of the form \( h = 2\tau = 1/N \) so that both conditions are satisfied. We apply the leapfrog scheme and RKS4 to the MOL approximation (48), and integrate from \( t = 0 \) to \( t = T \). We measure the errors of the schemes by using the discrete \( L_2 \)-norm
\[
\varepsilon_{u,L2} = \max_{0 < n \leq 2NT} ||\varepsilon_n||, \quad \varepsilon_{v,L2} = \max_{0 < n \leq 2NT} ||\varepsilon_{n+1/2}||,
\]
the discrete energy norm
\[
\varepsilon_e = \max_{0 < n \leq 2NT} ||(\varepsilon_n, \varepsilon_{n+1/2})||_{W_h}
\]
and maximum norm
\[
\varepsilon_{u,\max} = \max_{0 < n \leq 2NT} \{ ||\varepsilon_n||_\infty \}, \quad \varepsilon_{v,\max} = \max_{0 < n \leq 2NT} \{ ||\varepsilon_{n+1/2}||_\infty \}
\]
with \( || \cdot ||_\infty \) the maximum norm on \( \mathbb{R}^m \).

Table 1: Numerical results for (47) using the leapfrog scheme

<table>
<thead>
<tr>
<th>( N )</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
<th>640</th>
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<td>18.15</td>
<td>20.17</td>
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<td>2.01</td>
<td>2.00</td>
<td>2.00</td>
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<tr>
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<td>2.01</td>
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</tr>
<tr>
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<td>17.66</td>
<td>19.68</td>
<td>21.69</td>
<td>23.69</td>
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<tr>
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<td>2.02</td>
<td>2.01</td>
<td>2.00</td>
<td>2.00</td>
<td>2.00</td>
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<tr>
<td>( -\log_2 \varepsilon_{v,\max} )</td>
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<td>15.75</td>
<td>17.76</td>
<td>19.77</td>
<td>21.77</td>
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<tr>
<td>( -\log_2 \varepsilon_e )</td>
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Table 2: Numerical results for (47) using RKS4

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<th>80</th>
<th>160</th>
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<th>640</th>
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<td>(-\log_2 \varepsilon_{u,L2})</td>
<td>19.17</td>
<td>23.16</td>
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<td>35.15</td>
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<td>4.00</td>
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<tr>
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<td>29.67</td>
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<td>30.70</td>
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<tr>
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<td>28.59</td>
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<td>3.15</td>
<td>2.80</td>
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</table>

Table 1 and Table 2 show that the observed order of the leapfrog scheme and RKS4 is more than or equal 2. We observe that the order for \(u\) of RKS4 is higher than expected results from Theorem 4.1.

References


