On Mean-Square and Asymptotic Stability for Numerical Approximations of Stochastic Ordinary Differential Equations

Rózsa Horváth–Bokor* and Taketomo Mitsui†

Abstract

This note tries to connect the stochastic mean-square stability [Saito and Mitsui, SIAM J. Numer.Anal., 33(1996), pp. 2254-2267] and the asymptotic stability [D.J. Higham, SIAM J. Numer. Anal., 38(2000), pp. 753-769]. Saito and Mitsui generalizes the deterministic A-stability for a stochastic differential equation test problem with multiplicative noise. For the test equation, we know that the asymptotic stability in the mean-square sense implies the stochastic asymptotic stability in large. We will prove the same property for numerical schemes.

A.M.S. Classification codes (1991): 60H10, 65C05, 65C20, 65U05

Keywords: Stochastic ordinary differential equations, numerical methods, mean square convergence

1 Introduction

We consider the scalar stochastic initial value problem (SIVP) for the Itô ordinary differential equations (SODE) given by :

$$\mathrm{d}X(t) = a(t, X(t))\mathrm{d}t + b(t, X(t))\mathrm{d}W(t) \quad \text{for} \quad 0 \leq t \leq T \quad \text{and} \quad X(t) \in \mathbb{R}, \tag{1}$$

where functions are defined by

$$a : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, \quad b : [0, +\infty) \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad X(0) = X_0,$$

and $W(t)$ is the 1-dimensional Brownian motion. Let $\mathcal{F}_t$ denote the increasing family of σ-algebras (filtration) generated by the Brownian motion $W(s), s \leq t$. Details about this stochastic object and corresponding calculus can be found in [1, 4].

We consider an Itô equation (1) with a steady solution $X_t \equiv 0$. This means that $a(t, 0) = b(t, 0) = 0$ holds. $X^{t_0,0}(t)$ means that $X^{t_0,0}(t_0) = 0$.

The following definitions are due to Hasminski:

*Department of Mathematics and Computing University of Veszprém, e–mail: hrozsa@almos.vein.hu

†Graduate School of Information Science, Nagoya University, e-mail: tom.mitsui@cc.nagoya-u.ac.jp

Definition 1 *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of (1) is said to be* **stochastically stable** *if for any* $\epsilon > 0$ *and* $t_0 \geq 0$

$$\lim_{x_0 \to 0} P(\sup_{t \geq t_0} \| X^{t_0,x_0}(t) \| \geq \epsilon) = 0.$$

Definition 2 *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of (1) is said to be* **stochastically asymptotically stable** *if, in addition to being stochastically stable,*

$$\lim_{x_0 \to 0} P(\lim_{t \to +\infty} \| X^{t_0,x_0}(t) \| = 0) = 1.$$

Definition 3 *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of (1) is said to be* **stochastically asymptotically stable in the large**, *if moreover to the above two,*

$$P(\lim_{t \to +\infty} \| X^{t_0,x_0}(t) \| = 0) = 1. \text{ for all } x_0.$$

For the general SODE (1) Kloeden and Platen (cf. [5]) gave the following definition:

Definition 4 *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of (1) is* **stable in the** *p-th* **mean** *if for*

$$\forall \epsilon > 0 \ \forall t_0 > 0, \ \exists \delta = \delta(t_0, \epsilon) > 0$$

such that

$$E \| X^{t_0,x_0}(t) \|^p < \epsilon$$

for all $t \geq t_0$ *and* $\| x_0 \| < \delta$.

Definition 5 *The steady state solution* $X^{t_0,0}(t) \equiv 0$ *of (1) is* **asymptotically stable in the** *p-th* **mean** *if in addititon, there exists a* $\delta_0 = \delta(t_0)$ *such that*

$$\lim_{t \to +\infty} E \| X^{t_0,x_0}(t) \|^p = 0 \text{ for all } \| x_0 \| < \delta_0.$$

The most frequently used case $p = 2$ is called the mean-square case and in the sequel we focus our investigation on the mean-square stability.

We suppose that the equation (1) has a unique, bounded strong solution $X(t)$ in the mean-square sense.

Usual deterministic time discretization of a bounded time-interval $[0, T], \ T > 0$ is of the form $0 = t_0 < t_1 < \ldots < t_N = T$, where N is a natural number. We suppose that $t_0 = 0$ and adopt an equidistant discretization with the step size $\Delta = t_{n+1} - t_n, \ n = 0, 1, \ldots, N-1$.

In the sequel Y_n always denotes the approximation of $X(t_n)$ using a given numerical scheme with the step size Δ.

2 Mean-square and asymptotic stabilities for the test equation

Consider the one-dimensional stochastic test equation

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t) \qquad (t \geq 0), \quad X(0) = x_0, \tag{2}$$

where λ and μ are complex numbers and we assume that x_0 is a non zero deterministic value.

The exact solution of (2) is

$$X(t) = \exp\left((\lambda - \frac{\mu^2}{2})t + \mu W(t)\right) x_0,$$

which is sometimes called geometric Brownian motion. It has the second moment

$$E|X(t)|^2 = \exp((2\Re(\lambda) + |\mu|^2)t)|x_0|^2.$$

For this test equation, the stationary solution $X(t) \equiv 0$ is stochastically asymptotically stable in large if the inequality

$$\Re(\lambda - \frac{1}{2}\mu^2) < 0$$

holds. Saito-Mitsui [6] showed that the zero solution of equation (2) is asymptotically mean-square stable if and only if

$$2\Re(\lambda) + |\mu|^2 < 0,$$

as we can see the form of the second moment of $X(t)$.

Note that since the inequality $\Re(2\lambda - \mu^2) \leq 2\Re(\lambda) + |\mu|^2$ is always valid, the asymptotic stability in the mean-square sense implies the stochastic asymptotic stability in large.

As considered in [2] and [6] the analytical theory is not applicable to numerical schemes because it is impossible to carry out a numerical scheme until all the sample paths (2) diminish to zero if $\Re(\lambda) > 0$ and $\Re(\lambda - \frac{1}{2}\mu^2) < 0$.

Generally, in the vector case, when we apply a numerical scheme (Y_n) to the equation (2) and take the mean-square norm, we obtain a one-step difference equation of the form

$$E \parallel Y_{n+1} \parallel^2 = R(\overline{\Delta}, k) E \parallel Y_n \parallel^2, \tag{3}$$

where $\overline{\Delta} = \Delta\lambda$ and $k = -\mu^2/\lambda$.

Saito and Mitsui in their work [6] called the function $R(\overline{\Delta}, k)$ as the stability function of the scheme. In this case $E \parallel Y_n \parallel^2 \to 0$ as $n \to +\infty$ iff

$$|R(\overline{\Delta}, k)| < 1.$$

They gave in their work [6] the following definition:

Definition 6 *The scheme is said to be MS-stable for those values of $\overline{\Delta}$ and k satisfying*

$$|R(\overline{\Delta}, k)| < 1. \tag{4}$$

The set $\mathcal{R}$ given by

$$\mathcal{R} = \{(\overline{\Delta}, k) : |R(\overline{\Delta}, k)| < 1\}$$

is analogously called the domain of MS-stability of the scheme.

HIGHAM [3] introduces the new concept of stability.

Definition 7 *When a numerical scheme is applied to the stohastically asymptotically stable in large as equation (2) and generating the sequence (Y_n), it is said to be numerically asymptotically stable in large (or simply, asymptotically stable) if*

$$\lim_{n\to+\infty} \| Y_n \| = 0 \tag{5}$$

with probability 1.

We can establish a relationship between two stability concepts as follows.

Theorem 1 *The MS-stable schemes satisfying (4) are numerically asymptotically stable in large.*

In the proof of Theorem we will use the statement of the following Lemma.

Lemma 1

$$\lim_{n\to+\infty} \| Y_n \| = 0 \ \textit{(a.s)}$$

iff

$$\lim_{n\to+\infty} P(\cup_{k=n}^{\infty}\{\| Y_k \| \geq \epsilon\}) = 0 \quad \textit{for all} \quad \epsilon > 0. \tag{6}$$

Proof of Lemma. We prove this statement by defining for $n \in \mathbb{N}$ and $\epsilon > 0$ the events

$$A_n(\epsilon) = \{\| Y_n \| \geq \epsilon\}$$

and

$$A(\epsilon) = \overline{\lim_n} A_n(\epsilon) = \cap_{n=1}^{+\infty} \cup_{k=n}^{+\infty} A_k(\epsilon).$$

We notice that since $\cup_{k=n}^{+\infty} A_k(\epsilon) \downarrow A(\epsilon)$ as $n \to +\infty$, the equality $P(A(\epsilon)) = \lim_{n\to+\infty} P(\cup_{k=n}^{+\infty} A_k(\epsilon))$ holds.

Now we define an event

$$D = \{\omega \in \Omega; \| Y_n \| \text{ does not converge to 0 as } n \to +\infty\}.$$

Then

$$D = \cup_{\epsilon>0} A(\epsilon) = \cup_{m=1}^{+\infty} A\left(\frac{1}{m}\right),$$

because of $A(\epsilon_1) \subset A(\epsilon_2)$ for $\epsilon_1 > \epsilon_2$.

The above equalities imply that $\lim_{n\to+\infty} \| Y_n \| = 0$ with probability 1 that is equivalent to $P(D) = 0$. It holds only if and only if $P(A(\epsilon)) = 0$ for all $\epsilon > 0$, which is equivalent to

$$\lim_{n\to+\infty} P(\cup_{k=n}^{\infty}\{\| Y_k \| \geq \epsilon\}) = 0 \quad \text{for all} \quad \epsilon > 0. \tag{7}$$

∎

Using this result we can prove the statement of the Theorem.

Proof of Theorem.

We estimate the right-hand side of the inequality (7)

$$\begin{aligned} P(\cup_{l=n}^{\infty}\{\| Y_l \| \geq \epsilon\}) &\leq \sum_{l=n}^{\infty} P(\| Y_l \| \geq \epsilon) \leq \sum_{l=n}^{\infty} \frac{E \| Y_l \|^2}{\epsilon^2} \\ &\leq \sum_{l=n}^{\infty} \frac{R(\overline{\Delta}, k)^l E \| Y_0 \|^2}{\epsilon^2} \leq \frac{E \| Y_0 \|^2}{\epsilon^2} \sum_{l=n}^{\infty} R(\overline{\Delta}, k)^l \end{aligned}$$

tends to 0 as $n \to +\infty$ due to the condition (4) for all $\epsilon > 0$. Now we attain

$$\lim_{n\to\infty} \| Y_n \| = 0 \quad \text{(a.s).}$$

∎

By this we have the same conclusion as for exact solution: all MS-stable schemes are asymptotically stable in large.

References

[1] L. Arnold, *Stochastic Differential Equations,* Wiley, New York, 1974.

[2] K. Burrage, P. M. Burrage and T. Mitsui, Numerical Solutions of Stochastic Differential Equations– Implementation and Stability Issues –,*J. Comput. Appl. Math.* 125, No. 1-2(2000),171-182. -

[3] D. J. Higham, Mean-Square and Asymptotic Stability of Numerical Methods for Stochastic Ordinary Differential Equations,*J. Numer. Anal.* 38 (2000), 753-769.

[4] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus,* Springer-Verlag, (Berlin Heidelberg New York, 1988).

[5] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations,* Springer-Verlag, (Berlin Heidelberg New York, 1992).

[6] Y. Saito and T. Mitsui, Stability Analysis of Numerical Schemes for Stochastic Differential Equations, *SIAM J. Numer. Anal.* 33 (1996), 2254–2267.