DISTRIBUTION RELATION FOR BERNOULLI POLYNOMIALS ATTACHED TO FORMAL GROUP

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Abstract. We extend a well-known distribution relation for ordinary Bernoulli polynomials to that of Bernoulli polynomials attached to formal group.

Let q be an indeterminate and let $\mathfrak{o}$ be the formal power series ring in $q-1$ over some $\mathbb{Q}$-algebra. Furthermore let $\mathfrak{F}(X,Y)$ be a 1-dimensional commutative formal group defined over $\mathfrak{o}$ and let $\mathfrak{f}(X)$ be an isomorphism from the additive formal group $X+Y$ to $\mathfrak{F}(X,Y)$. We note that there exists a unique isomorphism $\mathfrak{f}_{\mathfrak{F}}(X)$ from $X+Y$ to $\mathfrak{F}(X,Y)$ defined over $\mathfrak{o}$ such that $\mathfrak{f}'_{\mathfrak{F}}(0)=1$. And $\mathfrak{f}(X)$ is equal to $\mathfrak{f}_{\mathfrak{F}}(cX)$ for some invertible element $c \in \mathfrak{o}^{\times}$. Conversely for any $c \in \mathfrak{o}^{\times}$, $\mathfrak{f}_{\mathfrak{F}}(cX)$ is an isomorphism from $X+Y$ to $\mathfrak{F}(X,Y)$. We note that we also denote $\mathfrak{f}_{\mathfrak{F}}(cX)$ by $[X]_{\mathfrak{F},c}$. Throughout this paper we assume that

Assumption 1.

$$\mathrm{ord}_{q-1}\mathfrak{f}_{\mathfrak{F}}^{(n)}(0) \geq n-1 \quad \textit{for all } n \geq 1,$$

where $\mathfrak{f}_{\mathfrak{F}}^{(n)}$ *means the* n*-th derivative of* $\mathfrak{f}_{\mathfrak{F}}$*.*

We note that by this assumption $\mathfrak{F}_n(A,B)$ (see Definition 1 below) and $\mathfrak{f}(a)$ are convergent in $\mathfrak{o}$ for any $A, B \in \mathfrak{o}[[X]]$ and $a \in \mathfrak{o}$ as formal power series (see [6, Remark 3]).

Definition 1. *For each non-negative integer* n *we denote the expansion of* $\mathfrak{F}(X,Y)^n$ *by*

$$\mathfrak{F}(X,Y)^n = \sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} X^i Y^j \ ,$$

and we set

$$\mathfrak{F}_n(A,B) = \sum_{i,j\geq 0} \binom{n}{i,j}_{\mathfrak{F}} a_i b_j$$

for any power series $A = \sum_{n\geq 0} a_n \frac{X^n}{n!}$ *and* $B = \sum_{n\geq 0} b_n \frac{X^n}{n!}$ *in* $\mathfrak{o}[[X]]$*. And we define the* $*_{\mathfrak{F}}$*-product by*

$$A *_{\mathfrak{F}} B = \sum_{n\geq 0} \mathfrak{F}_n(A,B) \frac{X^n}{n!} \ .$$

We note that we also denote $\mathfrak{F}_n(A,B)$ by $(a +_{\mathfrak{F}} b)^n$.

We can prove $(\mathfrak{o}[[X]], +, *_{\mathfrak{F}})$ is an $\mathfrak{o}$-algebra (see [6, Proposition 1]). Next we extend the following map:

$$X^n \mapsto c^n \underbrace{X *_{\mathfrak{F}} \cdots *_{\mathfrak{F}} X}_{n \text{ times}}$$

$\mathfrak{o}$-linearly. Hence we can get a natural homomorphism from $(\mathfrak{o}[[X]],+,\cdot)$ to $(\mathfrak{o}[[X]],+,*_{\mathfrak{F}})$. We denote the image of $A\in\mathfrak{o}[[X]]$ under this homomorphism by $A_{\mathfrak{F},c}$. And we define an analogue of power series A attached to $\mathfrak{F}$ and c by $A_{\mathfrak{F},c}$. Next we define Bernoulli numbers $\beta_n(\mathfrak{F},c)$ and Bernoulli polynomials $\beta_n(t;\mathfrak{F},c)$ attached to $\mathfrak{F}$ and c as follows:

Definition 2. *For each non-negative integer n we define the n-th Bernoulli number $\beta_n(\mathfrak{F},c)$ and Bernoulli polynomial $\beta_n(t;\mathfrak{F},c)$ attached to $\mathfrak{F}(X,Y)\in\mathfrak{o}[[X,Y]]$ and $c\in\mathfrak{o}^\times$ by the coefficient of $\frac{X^n}{n!}$ in $\left(\frac{X}{e^X-1}\right)_{\mathfrak{F},c}$ and that of $\frac{X^n}{n!}$ in $\left(\frac{Xe^{tX}}{e^X-1}\right)_{\mathfrak{F},c}$ respectively.*

We note that $\beta_n(0;\mathfrak{F},c)=\beta_n(\mathfrak{F},c)$. And because of

$$\begin{aligned}\left(\frac{Xe^{tX}}{e^X-1}\right)_{\mathfrak{F},c}&=\left(\frac{X}{e^X-1}\right)_{\mathfrak{F},c}*_{\mathfrak{F}}(e^{tX})_{\mathfrak{F},c}\\&=\left(\frac{X}{e^X-1}\right)_{\mathfrak{F},c}*_{\mathfrak{F}}e^{[t]_{\mathfrak{F},c}X}\\&=\sum_{n\geq0}\beta_n(\mathfrak{F},c)\frac{X^n}{n!}*_{\mathfrak{F}}\sum_{n\geq0}[t]^n_{\mathfrak{F},c}\frac{X^n}{n!},\end{aligned}$$

we can denote $\beta_n(t;\mathfrak{F},c)$ by $(\beta(\mathfrak{F},c)+_{\mathfrak{F}}[t]_{\mathfrak{F}})^n$.

To state our main result, we need to define a new formal group.

Definition 3. *For any $a\in\mathfrak{o}^\times$ we define new 1-dimensional commutative formal group over $\mathfrak{o}$ by*

$$\mathfrak{F}_{(a)}(X,Y)=\frac{\mathfrak{F}([a]_{\mathfrak{F},c}X,[a]_{\mathfrak{F},c}Y)}{[a]_{\mathfrak{F},c}}.$$

Let $[X]_{\mathfrak{F}_{(a)},d}$ be an isomorphism from the additive formal group $X+Y$ to $\mathfrak{F}_{(a)}(X,Y)$ for $d\in\mathfrak{o}^\times$.

Lemma 1. *If $d=\frac{a}{[a]_{\mathfrak{F},c}}c$, then we have*

$$[X]_{\mathfrak{F}_{(a)},d}=\frac{[aX]_{\mathfrak{F},c}}{[a]_{\mathfrak{F},c}}.$$

Proof. If we put $\mathfrak{g}=\frac{[aX]_{\mathfrak{F},c}}{[a]_{\mathfrak{F},c}}$, then $\mathfrak{g}$ is a power series in $\mathfrak{o}[[X]]$ and $\mathfrak{g}(0)=0,\ \mathfrak{g}'(0)=d$. Hence it is sufficient to prove that

$$\mathfrak{g}(\mathfrak{g}^{-1}(X)+\mathfrak{g}^{-1}(Y))=\mathfrak{F}_{(a)}(X,Y).$$

Now we again note $[X]_{\mathfrak{F},c}=\mathfrak{f}_{\mathfrak{F}}(cX)$. Because of

$$\mathfrak{g}^{-1}(X)=\frac{\mathfrak{f}_{\mathfrak{F}}^{-1}(\mathfrak{f}_{\mathfrak{F}}(ac)X)}{ac}\quad\text{and}\quad\mathfrak{F}(X,Y)=\mathfrak{f}_{\mathfrak{F}}(\mathfrak{f}_{\mathfrak{F}}^{-1}(X)+\mathfrak{f}_{\mathfrak{F}}^{-1}(Y)),$$

we have

$$\begin{aligned}\mathfrak{g}(\mathfrak{g}^{-1}(X)+\mathfrak{g}^{-1}(Y))&=\frac{\mathfrak{f}_{\mathfrak{F}}\left(ac\left(\frac{\mathfrak{f}_{\mathfrak{F}}^{-1}(\mathfrak{f}_{\mathfrak{F}}(ac)X)}{ac}+\frac{\mathfrak{f}_{\mathfrak{F}}^{-1}(\mathfrak{f}_{\mathfrak{F}}(ac)Y)}{ac}\right)\right)}{\mathfrak{f}_{\mathfrak{F}}(ac)}\\&=\frac{\mathfrak{f}_{\mathfrak{F}}(\mathfrak{f}_{\mathfrak{F}}^{-1}(\mathfrak{f}_{\mathfrak{F}}(ac)X)+\mathfrak{f}_{\mathfrak{F}}^{-1}(\mathfrak{f}_{\mathfrak{F}}(ac)Y)}{\mathfrak{f}_{\mathfrak{F}}(ac)}\\&=\frac{\mathfrak{F}([a]_{\mathfrak{F},c}X,[a]_{\mathfrak{F},c}Y)}{[a]_{\mathfrak{F},c}}\\&=\mathfrak{F}_{(a)}(X,Y).\end{aligned}$$

This completes the proof of Lemma 1 $\square$

Our main result of this paper is as follows:

Theorem 1. *For each non-negative integer n and positive integer m we have*

$$\beta_n(t;\mathfrak{F},c) = \frac{[m]_{\mathfrak{F},c}^n}{m}\sum_{l=0}^{m-1}\beta_n\left(\frac{t+l}{m};\mathfrak{F}_{(m)},d\right),$$

where $d = \frac{m}{[m]_{\mathfrak{F},c}}c$

The distribution relation for the ordinary Bernoulli polynomials $B_n(t)$ is

$$B_n(t) = m^{n-1}\sum_{l=0}^{m-1}B_n\left(\frac{t+l}{m}\right).$$

By considering the generating function of $B_n(t)$, the distribution relation is equivalent to

$$\frac{Xe^{tX}}{e^X-1} = \frac{1}{m}\sum_{l=0}^{m-1}\frac{mXe^{\frac{t+l}{m}mX}}{e^{mX}-1}$$

and this is a trivial equality. Hence if we apply our homomorphism to the above equality, we can get

$$\left(\frac{Xe^{tX}}{e^X-1}\right)_{\mathfrak{F},c} = \frac{1}{m}\sum_{l=0}^{m-1}\left(\frac{mXe^{\frac{t+l}{m}mX}}{e^{mX}-1}\right)_{\mathfrak{F},c}. \tag{1}$$

However in general it is not clear that the relationship of $A(X)_{\mathfrak{F},c}$ and $A(aX)_{\mathfrak{F},c}$ for $A\in\mathfrak{o}[[X]]$ and $a\in\mathfrak{o}$. Hence we provide several lemmas.

Lemma 2. *For any $A,\ B\in\mathfrak{o}[[X]]$ we have*

$$A([a]_{\mathfrak{F},c}X) *_{\mathfrak{F}} B([a]_{\mathfrak{F},c}X) = (A *_{\mathfrak{F}_{(a)}} B)([a]_{\mathfrak{F},c}X). \tag{2}$$

Proof. We put $A = \sum_{n\geq 0} a_n\frac{X^n}{n!}$ and $B = \sum_{n\geq 0} b_n\frac{X^n}{n!}$. By the definition of $*_{\mathfrak{F}}$-product, the left hand side of (2) is equal to

$$\begin{aligned} A([a]_{\mathfrak{F},c}X) *_{\mathfrak{F}} B([a]_{\mathfrak{F},c}X) &= \sum_{n\geq 0}\mathfrak{F}_n(A([a]_{\mathfrak{F},c}X), B([a]_{\mathfrak{F},c}X))\frac{X^n}{n!} \\ &= \sum_{n\geq 0}\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}}[a]_{\mathfrak{F},c}^{i+j}a_ib_j\frac{X^n}{n!}. \end{aligned}$$

On the other hand the right hand side of (2) is equal to

$$\begin{aligned} (A *_{\mathfrak{F}_{(a)}} B)([a]_{\mathfrak{F},c}X) &= \sum_{n\geq 0}\mathfrak{F}_{(a)n}(A,B)[a]_{\mathfrak{F},c}^n\frac{X^n}{n!} \\ &= \sum_{n\geq 0}\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}_{(a)}}a_ib_j[a]_{\mathfrak{F},c}^n\frac{X^n}{n!}. \end{aligned}$$

Now by the definition of $\mathfrak{F}_{(a)}$ and $\binom{n}{i,j}_{\mathfrak{F}_{(a)}}$, we have

$$\begin{aligned} \sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}_{(a)}}X^iY^j &= \mathfrak{F}_{(a)}(X,Y)^n \\ &= \frac{\mathfrak{F}([a]_{\mathfrak{F},c}X, [a]_{\mathfrak{F},c}Y)^n}{[a]_{\mathfrak{F},c}^n} \\ &= \frac{1}{[a]_{\mathfrak{F},c}^n}\sum_{i,j\geq 0}\binom{n}{i,j}_{\mathfrak{F}}[a]_{\mathfrak{F},c}^{i+j}X^iY^j. \end{aligned}$$

Hence we have

$$\binom{n}{i,j}_{\mathfrak{F}}[a]_{\mathfrak{F},c}^{i+j}=\binom{n}{i,j}_{\mathfrak{F}_{(a)}}[a]_{\mathfrak{F},c}^{n}.$$

This completes the proof of Lemma 2. $\square$

Lemma 3. *Let* $d=\frac{a}{[a]_{\mathfrak{F},c}}c$. *For any* $a\in\mathfrak{o}$ *and* $A\in\mathfrak{o}[[X]]$ *we have*

$$A(aX)_{\mathfrak{F},c}=A_{\mathfrak{F}_{(a)},d}([a]_{\mathfrak{F},c}X).$$

Proof. It is sufficient to prove for $A=X^n$. By Lemma 2 and the mathematical induction, we have

$$\underbrace{[a]_{\mathfrak{F},c}X*_{\mathfrak{F}}\cdots*_{\mathfrak{F}}[a]_{\mathfrak{F},c}X}_{n\text{ times}}=\underbrace{X*_{\mathfrak{F}_{(a)}}\cdots*_{\mathfrak{F}_{(a)}}X}_{n\text{ times}}([a]_{\mathfrak{F},c}X).$$

This means

$$(aX)^n_{\mathfrak{F},c}=(X^n)_{\mathfrak{F}_{(a)},d}([a]_{\mathfrak{F},c}(X))$$

and completes the proof of Lemma 3. $\square$

Now we return to the proof of theorem. Let $B(t;X)$ be the generating function of Bernoulli polynomials, i.e. ,

$$B(t;X)=\frac{Xe^{tX}}{e^X-1}.$$

By (1) and Lemma 3 we have

$$\begin{aligned}B_{\mathfrak{F},c}(t;X)&=\frac{1}{m}\sum_{l=0}^{m-1}\left(\frac{mXe^{\frac{t+l}{m}mX}}{e^{mX}-1}\right)_{\mathfrak{F},c}\\&=\frac{1}{m}\sum_{l=0}^{m-1}B\left(\frac{t+l}{m};mX\right)_{\mathfrak{F},c}\\&=\frac{1}{m}\sum_{l=0}^{m-1}B_{\mathfrak{F}_{(m)},d}\left(\frac{t+l}{m};[m]_{\mathfrak{F},c}X\right).\end{aligned}$$

By comparing the coefficient of $\frac{X^n}{n!}$, we cat get what we want.

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