On Silver's Remark on the Ramseyness of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Sigma}^1_2$

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Summary

Our topic in this paper is subsets of reals ($\subseteq {}^\omega 2$) and their Ramseyness.

We think of ${}^\omega 2$ as the product topological space of ω-many discrete spaces $\{0, 1\}$, which is a perfect Polish space, that is, a complete separable metric space without isolated points.

For $X \subseteq \omega$, $[X]^\omega$ denotes the set of infinite subsets of X, and $[X]^{<\omega}$ denotes the set of finite subsets of X.

$A \subseteq {}^\omega 2$ is said to be *Ramsey* when there is some infinite $S \subseteq \omega$ such that either $[S]^\omega \subseteq A$ or $[S]^\omega \cap A$ is empty. If this holds, S is said to be homogeneous for A.

$A \subseteq {}^\omega 2$ is said to be *completely Ramsey* when for every continuous function $f : {}^\omega 2 \to {}^\omega 2$, $f^{-1}(A)$, the inverse image of A, is Ramsey. For $s \in [\omega]^{<\omega}$ and $S \in [\omega]^\omega$ with $\min s < \max S$, $[s, S]$ denotes $\{X \in [\omega]^\omega \mid s \subseteq X \subseteq s \cup S\}$. Then if A is completely Ramsey and s, S are as above, there is $S' \subseteq S$ such that $[s, S'] \subseteq A$ or $[s, S'] \cap A = 0$.

Erdös and Rado [1] showed the existence of non-Ramsey subsets using the Axiom of Choice (AC). Mathias showed [4] that the use of AC is necessary. Thus this non-Ramsey subset is non-constructible. It is natural to ask how complex subsets must be to be Ramsey.

Galvin and Prikry [3] showed that every Borel subset is completely Ramsey.

J. Silver [2] proved the following:

1. Every $\mathbf{\Sigma}^1_1$ subset of ${}^\omega 2$ is Ramsey.

2. If there is a measurable cardinal then every $\mathbf{\Sigma}^1_2$ subset of ${}^\omega 2$ is also Ramsey.

In his paper Silver used the forcing notion making $\mathrm{MA}(\omega_1)$ and remarked that instead of this forcing we can use a less complicated forcing notion.

In this paper we give the definitions of these concepts and a more detailed proof of Silver's remark. We argue as follows in the case of $\mathbf{\Sigma}^1_1$: Let A be a $\mathbf{\Sigma}^1_1$ set. It is known that every $\mathbf{\Sigma}^1_1$ set is split into $\aleph_1$ Borel sets in a canonical way. We write this situation as $A = \bigcup_{\alpha<\omega_1} A_\alpha$. We start from a countable transitive model N of a sufficiently large fragment of ZFC. Next we show that there exists a p.o. set $P_{\mathcal{F}} \in N$ such that if G is any $P_{\mathcal{F}}$-generic over N, then in $N[G]$ there is a generic $X \in [\omega]^\omega$ such that each A_α is open in $[X]^\omega$. Then by the second countability axiom, there exists $\gamma < \omega_1$ such that

$$\bigcup_{\alpha<\gamma} (A_\alpha \cap [X]^\omega) = A \cap [X]^\omega$$

and this is open in $[X]^\omega$. Using Galvin-Prikry's results, there exists $Y \subseteq X$ such that

$$[Y]^\omega \subseteq \bigcup_{\alpha<\gamma} A_\alpha \text{ , or}$$
$$[Y]^\omega \cap A = \emptyset$$

In fact, it is in $N[G]$ that this homogeneous situation holds. But this is described as a Π^1_1 statement. Since it can be assumed that Π^1_1 is absolute for $N[G]$, this holds in V, too. Thus A is Ramsey.

Following this method of proof, we can show that

1. Every $\mathbf{\Sigma}^1_1$ subset of ${}^\omega 2$ is completely Ramsey.
2. Every provably $\mathbf{\Delta}^1_2$ subset of ${}^\omega 2$ is completely Ramsey.
3. For $a \subseteq \omega$, if $\omega_1^{L[a]} < \omega_1$ then every $\Sigma^1_2(a)$ subset of ${}^\omega 2$ is Ramsey.

We prove the first result to demonstrate our method of proof and the difference between Silver's and ours.

Then, by a similar proof we extend the first result to provably $\mathbf{\Delta}^1_2$ subsets.

Finally, the simplicity of our forcing notion enables us to find the weakest cardinality hypothesis and to effectivize Silver's result.

These results are optimal in the following sense: Since ZF+"$V = L$" implies the existence of a Δ^1_2 non-Ramsey set, we cannot prove Δ^1_2's Ramseyness from ZF only. Similarly, ZF+"$V = L[\mu]$" (the universe is the inner model of measurability) implies the existence of a Δ^1_3 non-Ramsey set. So we cannot prove Δ^1_3's Ramseyness even from ZF+"$\exists$measurable".

Moreover note that our theorems are proved completely within ZF. Silver's paper did not show that AC is not necessary.

References

[1] P. Erdös and R. Rado, *Combinatorial theorems on classifications of subsets of a given set*, Proceedings of the London Mathematical Society, vol. 2 (1952), pp. 417–439.

[2] J. H. Silver, *Every analytic set is Ramsey*, Journal of Symbolic Logic, vol. 35 (1970), pp. 60–64.

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[4] A. R. D. Mathias, *Happy families*, Annals of Mathematical Logic, vol. 12 (1977), pp. 59–111.

[5] A. Kanamori, *The Higher Infinite*, Springer, Berlin, 1994.