On Silver's Remark on the Ramseyness of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Sigma}^1_2$

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Abstract

Our topic in this paper is subsets of reals ($\subseteq {}^{\omega}2$) and their Ramseyness.

Erdös and Rado [1, Example 1, p. 434] showed the existence of non-Ramsey subsets using the Axiom of Choice (AC). Mathias [4] showed that the use of AC is essential. It is natural to ask how complex subsets must be to be Ramsey.

${}^{\omega}2$ is a complete Polish space. Therefore we can develop descriptive set theory and can consider projective and analytical hierarchies.

Galvin and Prikry [3] showed that every Borel subset is Ramsey.

J. Silver [2] proved that every $\mathbf{\Sigma}^1_1$ subset of reals is Ramsey and that if there is a measurable cardinal then every $\mathbf{\Sigma}^1_2$ subset of reals is also Ramsey. In his proof Silver used the forcing notion making $\mathrm{MA}(\omega_1)$. He remarked that instead of this forcing we can use a less complicated forcing notion.

In this paper we give the definitions of these concepts and a more detailed proof of Silver's remark.

In the case of $\mathbf{\Sigma}^1_1$ our theorem is the same as Silver's result but our proof is a little different. Next we show that every provably $\mathbf{\Delta}^1_2$ subset is completely Ramsey by a similar proof. Finally, in the case of $\mathbf{\Sigma}^1_2$, the simplicity of our forcing notion enables us to find the weakest cardinality hypothesis. The last two results extend Silver's results. Throughout these three theorems we do not use AC, so they are completely within ZF.

1 Notation and Definition

ZF denotes Zermelo-Fraenkel set theory without the Axiom of Choice. The Axiom of Choice is referred to as AC. ZFC means ZF+AC. We assume ZF for the ambient theory if not specified otherwise.

ω is the set of natural numbers and is the least infinite ordinal. ω_1 is the set of at most countable ordinals and is the least uncountable ordinal. 0 denotes the empty set.

For sets X and Y, $\mathcal{P}(X)$ is the power set of X and ${}^Y X$ is the set of functions from Y into X. When Y is a natural number n, ${}^n X$ is identified with n-tuples of elements from X. If X is a set of ordinals, $[X]^\omega$ denotes the set of subsets of X whose order type is ω. Similarly, $[X]^{<\omega}$ denotes the set of subsets of X whose order type is finite. For example, $\mathcal{P}(\omega)$ is the disjoint union of $[\omega]^{<\omega}$ and $[\omega]^\omega$.

${}^\omega 2 = {}^\omega\{0,1\}$ is identified with $\mathcal{P}(\omega)$ through the characteristic function. Inclusion symbols $\subseteq$'s are used in this sense. We will study a property of subsets of reals ($X \subseteq {}^\omega 2$).

${}^\omega 2$ can be regarded as the topological product space of ω-many discrete spaces $\{0,1\}$. Open sets in this topology are generated from $\{x|n \in x\}_{n\in\omega}$ and $\{x|n \notin x\}_{n\in\omega}$ by finite intersection and arbitrary union. Throughout this paper, we fix this topology of ${}^\omega 2$.

As our topology of ${}^\omega 2$ makes a perfect Polish space we can develop descriptive set theory.

First we define the analytical hierarchy. For this we consider a second-order arithmetic structure

$$\mathcal{A}(a) = \langle \omega, {}^\omega 2, \in, +, \times, \exp, <, 0, 1, a \rangle,$$

where $a \in {}^\omega 2$ is thought of as an interpretation of a predicate over ω, $\exp(x,y)$ is x^y and second-order equality "$=^2$" is not allowed. Quantifiers bounded in ω are abbreviated to quantifiers with a superscript "0", and in ${}^\omega 2$ with a superscript "1". For example, "$\exists^1 y$" means "$\exists y \in {}^\omega 2$".

Definable subsets of ${}^r\omega \times {}^k({}^\omega 2)$ in this structure are classified by the difficulty of the formula defining them.

$A \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is Δ^0_0 *in* a *iff* A is definable in $\mathcal{A}(a)$ by a formula whose only quantifiers are bounded, i.e., $(\exists^0 n < t)$ or $(\forall^0 n < t)$, where t is a term representing numbers.

$A \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is *arithmetical in* a *iff* A is definable in $\mathcal{A}(a)$ by a formula whose only quantifiers are all numerical, i.e., $\exists^0$ or $\forall^0$. To a formula defining arithmetical relation, replacement and contraction of quantifiers are applied based on the following equivalences:

$$\begin{aligned} (\forall^0 p < q)\exists^0 m R(p,q,m,\mathbf{w}) \quad &\textit{iff} \quad \exists^0 m (\forall^0 p < q) R(p,q,(m)_p,\mathbf{w}) \\ \exists^0 m \exists^0 p R(p,m,\mathbf{w}) \quad &\textit{iff} \quad \exists^0 m R((m)_0,(m)_1,\mathbf{w})\ , \end{aligned}$$

where $(m)_i$ denotes the i-th element of a finite sequence of natural numbers which is coded by m in the usual Δ^0_0 manner. Through these transformations every arithmetical relation $A \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is either of the following:

$$A \in \Sigma^0_n(a) \quad \textit{iff} \quad \forall \mathbf{w}\,(\mathbf{w} \in A \leftrightarrow \exists^0 m_1 \forall^0 m_2 \ldots Q m_n R(\mathbf{w}, m_1, m_2, \ldots, m_n))\ ,$$

$$A \in \Pi^0_n(a) \quad \textit{iff} \quad \forall \mathbf{w}\,(\mathbf{w} \in A \leftrightarrow \forall^0 m_1 \exists^0 m_2 \ldots Q m_n R(\mathbf{w}, m_1, m_2, \ldots, m_n))$$

for some $R \subseteq {}^{r+n}\omega \times {}^k({}^\omega 2)$ Δ^0_0 in a, where quantifiers bounding $m_1, \ldots, m_n$ are alternating between $\exists^0$ and $\forall^0$. $\Delta^0_n(a)$ denotes $\Sigma^0_n(a) \cap \Pi^0_n(a)$.

It can be easily shown that

$$\Sigma^0_n(a) \cup \Pi^0_n(a) \subset \Delta^0_{n+1}(a) = \Sigma^0_{n+1}(a) \cap \Pi^0_{n+1}(a) \text{ , for } n > 0.$$

We call this classification the *arithmetical hierarchy.*

We can check that A is *recursive in a iff* $A \in \Sigma^0_1(a) \cap \Pi^0_1(a) = \Delta^0_1(a)$.

$A \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is *analytical in a iff* A is definable in $\mathcal{A}(a)$ by a formula without any restriction of quantifiers. Through some transformations, every analytical relation $A \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is either of the following: (These transformations are not easy since we cannot use even the countable axiom of choice.)

$$A \in \Sigma^1_n(a) \quad \textit{iff} \quad \forall \mathbf{w}\,(\mathbf{w} \in A \leftrightarrow \exists^1 x_1 \forall^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, x_2, \ldots, x_n)) \text{ ,}$$
$$A \in \Pi^1_n(a) \quad \textit{iff} \quad \forall \mathbf{w}\,(\mathbf{w} \in A \leftrightarrow \forall^1 x_1 \exists^1 x_2 \ldots Q x_n R(\mathbf{w}, x_1, x_2, \ldots, x_n))$$

for some $R \subseteq {}^r\omega \times {}^{k+n}({}^\omega 2)$ arithmetical in a, where quantifiers bounding $x_1, \ldots, x_n$ are alternating between $\exists^1$ and $\forall^1$. $\Delta^1_n(a)$ denotes $\Sigma^1_n(a) \cap \Pi^1_n(a)$.

It can be easily shown that

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We call this classification the *analytical hierarchy.*

Throughout these arithmetical and analytical notations, if $a = 0$ then "(0)" is suppressed.

Next we define the projective hierarchy. While this is usually defined using the open sets of the topology of the space and the projection, here we adopt the following properties as their definitions. For $n > 0$ and $A \subseteq {}^r\omega \times {}^k({}^\omega 2)$,

$$A \in \boldsymbol{\Sigma}^1_n \quad \textit{iff} \quad A \text{ is } \Sigma^1_n(a) \text{ for some } a \in {}^\omega 2 \text{ ,}$$
$$A \in \boldsymbol{\Pi}^1_n \quad \textit{iff} \quad A \text{ is } \Pi^1_n(a) \text{ for some } a \in {}^\omega 2 \text{ ,}$$
$$A \in \boldsymbol{\Delta}^1_n \quad \textit{iff} \quad A \text{ is } \Delta^1_n(a) \text{ for some } a \in {}^\omega 2 \text{ .}$$

It is easily shown that

$$\boldsymbol{\Sigma}^1_n \cup \boldsymbol{\Pi}^1_n \subset \boldsymbol{\Delta}^1_{n+1} = \boldsymbol{\Sigma}^1_{n+1} \cap \boldsymbol{\Pi}^1_{n+1} \text{ , for } n > 0.$$

We present the following proposition:

Proposition 1 *For $a \in {}^\omega 2$, $n > 0$, $R \subseteq {}^r\omega \times {}^k({}^\omega 2)$ is $\Sigma^0_n(a)$ iff there is a Σ^0_n $S \subseteq {}^r\omega \times {}^{k+1}({}^\omega 2)$ such that $\forall \mathbf{w}(R(\mathbf{w}) \leftrightarrow S(\mathbf{w}, a))$. Analogous statements hold for Π^0_n, Σ^1_n, and Π^1_n.*

Note that the analogous statements for Δ^0_n and Δ^1_n do not hold.

Now we have defined hierarchies of subsets of reals in terms of descriptive set theory.

From here we define a property of subsets of reals, Ramseyness. If A is a subset of ${}^\omega 2$, $S \in [\omega]^\omega$ is said to be homogeneous for A when $[S]^\omega \subseteq A$ or $[S]^\omega \cap A = 0$. When A has some homogeneous set S, A is said to be *Ramsey.* A is *completely Ramsey iff* $f^{-1}(A)$ is Ramsey for every continuous function $f : {}^\omega 2 \to {}^\omega 2$. Obviously if A is completely Ramsey then A is Ramsey, since the identity function is continuous! This complete Ramseyness is by Galvin-Prikry [3].

We freely use the technical terms and notations of forcing. In particular, for any p.o. set P and any condition $p, q \in P$ and formula φ,

$$\begin{aligned} p \Vdash_P \varphi \quad & \textit{iff} \quad p \text{ forces } \varphi \text{ in a forcing by } P\ , \\ p \,||_P\, \varphi \quad & \textit{iff} \quad p \Vdash_P \varphi \text{ or } p \Vdash_P \neg\varphi\ , \\ p \leq_P q \quad & \textit{iff} \quad p \text{ extends or is a stronger condition than } q\ . \end{aligned}$$

The subscripts P are suppressed when clear from the context.

Next we define the Mathias forcing and closed intervals. The Mathias forcing p.o. set is

$$P_M \stackrel{\text{def}}{=} \{\langle s, S\rangle \in [\omega]^{<\omega} \times [\omega]^\omega \mid \max s < \min S\}$$

ordered by:

$$\langle s, S\rangle \leq_{P_M} \langle t, T\rangle \stackrel{\text{def}}{\Leftrightarrow} t \subseteq s \ \wedge\ S \cup (s \setminus t) \subseteq T\ .$$

For $\langle s, S\rangle \in P_M$, the corresponding closed interval is

$$[s, S] \stackrel{\text{def}}{=} \{X \in [\omega]^\omega \mid s \subseteq X \subseteq s \cup S\}\ .$$

Actually $[s, S]$ is closed in $[\omega]^\omega$ in the sense of relative topology. For $\langle s, S\rangle \in P_M$ and $A \subseteq {}^\omega 2$ we borrow terms of forcing, which we define as follows:

$$\begin{aligned} \langle s, S\rangle \textit{ forces } A \quad (\langle s, S\rangle \Vdash^R A) \quad & \stackrel{\text{def}}{\Leftrightarrow} \quad [s, S] \subseteq A\ , \\ \langle s, S\rangle \textit{ decides } A \quad (\langle s, S\rangle \,||^R\, A) \quad & \stackrel{\text{def}}{\Leftrightarrow} \quad \langle s, S\rangle \Vdash^R A \ \vee\ \langle s, S\rangle \Vdash^R ({}^\omega 2 \setminus A)\ . \end{aligned}$$

The superscripts R are from "in the real world" and these symbols must be distinguished from the ordinary forcing notions $\Vdash$ and $||$. It is easy to check that

$$A \text{ is Ramsey } \textit{ iff } \ \exists S \in [\omega]^\omega (\langle 0, S\rangle \,||^R\, A)$$

and that if A is completely Ramsey then

$$\forall \langle s, S\rangle \in P_M \exists \langle s, S'\rangle \in P_M\, [\langle s, S'\rangle \leq \langle s, S\rangle \ \wedge\ \langle s, S'\rangle \,||^R\, A]\ .$$

For the latter, note that if $e(n)$ denotes the n-th least member of S then $f : X \mapsto s \cup e``X$ is continuous. In fact, this latter property is the definition of complete Ramseyness adopted by Silver [2]. Our definition is strictly stronger than this property in the following sense: Suppose there is a non-Ramsey set, say A. Define a continuous injection $f : {}^{\omega}2 \to {}^{\omega}2$ by

$$f(X) = \bigcup_{n \in X} \{2n, 2n+1\} \ .$$

Then $f``A$ satisfies the above property, as $\langle s, S \cap \{2n\}_{n\in\omega}\rangle$ or $\langle s, S \cap \{2n+1\}_{n\in\omega}\rangle$ decides $({}^{\omega}2 \backslash f``A)$. On the other hand, this is not completely Ramsey in our sense, as $f^{-1}(f``A) = A$ is not Ramsey.

Under AC, we can prove the existence of non-Ramsey sets.

Proposition 2 *If* AC *holds, there exists a non-Ramsey set. Moreover if* ${}^{\omega}2 \subseteq L$*, then this set can be taken to be* Δ^1_2.

Proof of Proposition: If AC holds, there is a well-ordering $\prec$ of ${}^{\omega}2$. Here we define

$$A \stackrel{\text{def}}{=} \left\{X \in [\omega]^{\omega} \mid \forall^1 Y (Y \in [X]^{\omega} \to Y \not\prec X)\right\} \ .$$

There is no homogeneous $Z \in [\omega]^{\omega}$ for A: First the $\prec$-least element of $[Z]^{\omega}$ is always in A, thus $[Z]^{\omega} \cap A \neq 0$. But if $[Z]^{\omega}$ were to be a subset of A, the existence of a sequence of distinct infinite sets

$$Z_0 \subset Z_1 \subset \cdots \subset Z$$

would give us a descending chain of $\prec$, which contradicts the well-orderedness of $\prec$.

Next if ${}^{\omega}2 \subseteq L$, the above well-ordering is taken to be the L-order $<_L$. This order is Δ^1_2. Thus the above definition of A tells us that A is Π^1_2. Moreover using

$$\mathrm{IS}_L(x, y) \leftrightarrow \{(x)_i \mid i \in \omega\} = \{z \in {}^{\omega}2 \mid z <_L y\} \ ,$$

which is Σ^1_2 (see [5] for more), A is reformulated as

$$A = \left\{X \in [\omega]^{\omega} \mid \exists^1 Y [\mathrm{IS}_L(Y, X) \wedge \forall^0 i ((Y)_i \notin [X]^{\omega})]\right\} \ ,$$

which is Σ^1_2. $\square$

In this proposition, the use of AC is essential. Mathias [4] showed that the following theorem holds.

Proposition 3 *If* ZF+ *"there is an inaccessible cardinal" is consistent, then* ZF+ *"all subsets of reals are Ramsey" is consistent.*

On the other hand we can prove the Ramseyness of sets in the projective low hierarchy. Silver [2] showed that

1. Every $\mathbf{\Sigma}^1_1$ subset of ${}^\omega 2$ is Ramsey,
2. If there is a measurable cardinal, then every $\mathbf{\Sigma}^1_2$ subset of ${}^\omega 2$ is Ramsey.

In his paper he suggested the possibility of the use of a less complicated forcing notion than his, and his proof used freely but not essentially the axiom of choice.

Our aim in this paper is showing in ZF that:

1. Every $\mathbf{\Sigma}^1_1$ subset of ${}^\omega 2$ is completely Ramsey,
2. Every provably $\mathbf{\Delta}^1_2$ subset of ${}^\omega 2$ is completely Ramsey,
3. For $a \subseteq \omega$, if $\omega_1^{L[a]} < \omega_1$ then every $\Sigma^1_2(a)$ subset of ${}^\omega 2$ is Ramsey.

The first result is exactly the same as Silver's except that the proof does not require AC.

2 Preliminary for Descriptive Set Theory

We give here some properties in descriptive set theory needed in the following sections.

Arithmetical and analytical hierarchies for Baire space ${}^\omega\omega$ are defined similarly to those of ${}^\omega 2$ by the structure

$$\mathcal{A}'(a) = \langle \omega, {}^\omega\omega, \mathrm{ap}, +, \times, \exp, <, 0, 1, a \rangle.$$

First we mention a correlation between ${}^\omega\omega$ and ${}^\omega 2$. These spaces are complete Polish spaces, so they are known to be Borel isomorphic in a general context. But in fact this isomorphism can be given explicitly.

Lemma 1 *There is an isomorphism $\pi : {}^\omega\omega \to {}^\omega 2$ such that $n \in \pi(x)$ and $\pi^{-1}(y)(m) = n$ are both Δ^0_3 in ${}^\omega\omega$ and ${}^\omega 2$ respectively.*

This isomorphism can be made from two injections

$$\begin{aligned} f : {}^\omega\omega \ni x &\mapsto 0^{x(0)}1^{x(1)}0^{x(2)}1^{x(3)} \cdots \in {}^\omega 2 \text{ , and} \\ g : {}^\omega 2 \ni y &\mapsto y \in {}^\omega\omega \end{aligned}$$

by using Schroeder-Bernstein's method. It is long but straightforward to check that this isomorphism is written by a bi-Δ^0_3 relation.

The following lemmata are frequently used in the proof of our main theorems. These are proved as usual in the effective context of ${}^\omega\omega$, and are derived in our space ${}^\omega 2$ through the above isomorphism. They do not require AC. Refer to [5] for proofs.

Lemma 2 *If A is $\mathbf{\Delta}^1_1$ then A is Borel.*

Lemma 3 *There is a theorem σ_1 of* ZF *such that for any transitive (set or class) $\in$-model M of σ_1, every Σ^1_1 relation is absolute for M.*

Lemma 4 (Shoenfield) *There is a theorem σ_2 of* ZF *such that for any transitive (set or class) $\in$-models M of σ_2 with ${\omega_1}^V \subseteq M$, every Σ^1_2 relation is absolute for M.*

Here the class M versions of these theorems are theorem schemata. But even in this case, σ_1 or σ_2 is a single theorem, so models have only to satisfy a fragment of ZF in each case.

The proofs of these might seem to use the regularity of ω_1. But, in fact, we can rewrite the proofs using only the lack of mapping from countable objects onto ω_1.

Lemma 5 *Every $\mathbf{\Sigma}^1_1$ set is the canonical union of $\aleph_1$ Borel increasing sets. More formally, for every Σ^1_1 formula $\varphi(\mathbf{w})$ with only free variables $\mathbf{w} \in {}^k({}^\omega 2)$, there are Π^1_1 formulae $\theta_\varphi(\mathbf{w}, z)$ and $\chi_\varphi(\mathbf{w}, z)$ such that*

1. *If z codes some countable ordinal, then $\forall^1 \mathbf{w}\, (\theta_\varphi(\mathbf{w}, z) \leftrightarrow \neg\chi_\varphi(\mathbf{w}, z))$.*
2. *If z does not code any countable ordinal, then $\forall^1 \mathbf{w}\, (\neg\theta_\varphi(\mathbf{w}, z) \wedge \neg\chi_\varphi(\mathbf{w}, z))$.*
3. *If z_α, z_β codes α, β and $\alpha \leq \beta$, then $\forall^1 \mathbf{w}\, (\theta_\varphi(\mathbf{w}, z_\alpha) \to \theta_\varphi(\mathbf{w}, z_\beta))$.*
4. *$\varphi(\mathbf{w})$ iff $\exists^1 z \theta_\varphi(\mathbf{w}, z)$.*

This lemma is an effective version of the usual $\aleph_1$-Borel lemma, and in fact is a single theorem of ZF through the Gödelization of formulae. Using this lemma, we can split $\mathbf{\Sigma}^1_1$ set into $\aleph_1$ Borel sets: Let $\Sigma^1_1(a)$ set

$$A = \{x \in {}^\omega 2 \mid \varphi(x, a)\} .$$

By taking $\alpha = \beta$ in condition 3, we can define for $\alpha < \omega_1$,

$$A_\alpha = \{x \in {}^\omega 2 \mid \text{ for some and any } z \text{ coding } \alpha,\ \theta_\varphi(x, a, z)\} .$$

These sets are Borel by condition 1. By condition 3,

$$\alpha \leq \beta \longrightarrow A_\alpha \subseteq A_\beta .$$

Finally, by condition 4,

$$A = \bigcup_{\alpha < \omega_1} A_\alpha .$$

Moreover this splitting lemma holds for $\mathbf{\Sigma}^1_2$;

Lemma 6 *Every $\mathbf{\Sigma}^1_2$ set is the canonical union of $\aleph_1$ Borel increasing sets. More formally, for every Σ^1_2 formula $\varphi(\mathbf{w})$ with only free variables $\mathbf{w} \in {}^k({}^\omega 2)$, there are Π^1_1 formulae $\theta_\varphi(\mathbf{w}, z)$ and $\chi_\varphi(\mathbf{w}, z)$ such that*

1. *If z codes some countable ordinal, then $\forall^1 \mathbf{w}\, (\theta_\varphi(\mathbf{w}, z) \leftrightarrow \neg\chi_\varphi(\mathbf{w}, z))$.*
2. *If z does not code any countable ordinal, then $\forall^1 \mathbf{w}\, (\neg\theta_\varphi(\mathbf{w}, z) \wedge \neg\chi_\varphi(\mathbf{w}, z))$.*
3. *If z_α, z_β codes α, β and $\alpha \leq \beta$, then $\forall^1 \mathbf{w}\, (\theta_\varphi(\mathbf{w}, z_\alpha) \to \theta_\varphi(\mathbf{w}, z_\beta))$.*
4. *$\varphi(\mathbf{w})$ iff $\exists^1 z \theta_\varphi(\mathbf{w}, z)$.*

3 On $\mathbf{\Sigma}^1_1$ sets

We use the following theorem as a basis. A proof which uses a very weak version of the axiom of choice is found in [3].

Theorem 1 (Galvin-Prikry) (AC) *Every Borel set of ${}^\omega 2$ is completely Ramsey.*

Now we show the following theorem with a different proof from Silver's original proof in [2].

Theorem 2 (Silver) *Every $\mathbf{\Sigma}^1_1$ set of ${}^\omega 2$ is completely Ramsey.*

Proof: First we have only to show that every $\mathbf{\Sigma}^1_1$ set of ${}^\omega 2$ is Ramsey. For if A is $\mathbf{\Sigma}^1_1$ and $f : {}^\omega 2 \to {}^\omega 2$ is continuous then $f^{-1}(A)$ is $\mathbf{\Sigma}^1_1$, too.

Given a $\Sigma^1_1(a)$ set $A^V = \{x \mid \varphi(x, a)\}$, we shall show that this A^V is Ramsey. The superscript V suggests that this set is defined in V from φ.

Let θ_φ and χ_φ be as in the splitting lemma (Lemma 5). Then we can define for $\alpha < \omega_1$,

$$A^V_\alpha = \{x \mid \text{ for some (any) } z \text{ coding } \alpha,\, \theta_\varphi(x, a, z)\}\ .$$

And we know $A^V = \bigcup_{\alpha < \omega_1^V} A^V_\alpha$.

Now suppose N is a countable transitive model of a fragment of ZF+AC sufficiently large for this proof with $a \in N$.

We argue in N from here. A^N and A^N_α for $\alpha < \omega_1^N$ are defined from φ, θ_φ and a in N, too. N can be assumed to satisfy the splitting lemma, so that $A^N = \bigcup_{\alpha < \omega_1^N} A^N_\alpha$ and each A^N_α is Borel. Moreover N can be assumed to satisfy Theorem 1, so N thinks each A^N_α is completely Ramsey (here we need that N satisfies at least a weak version of AC).

Claim 2.1 *There is $\mathcal{F} \subseteq [\omega]^\omega$ such that*

1. $|\mathcal{F}| \leq \omega_1^N$,
2. $E, E' \in \mathcal{F} \longrightarrow E \cap E' \in \mathcal{F}$,
3. $|\omega \setminus E| < \omega \longrightarrow E \in \mathcal{F}$,
4. $\forall \alpha < \omega_1 \forall s \in [\omega]^{<\omega} \exists U \in \mathcal{F} \, [\langle s, U \rangle \; ||^R \; A_\alpha^N]$.

Proof of Claim: Fix a sequence $\langle \alpha_\beta, s_\beta \rangle_{\beta < \omega_1}$ enumerating $\omega_1 \times [\omega]^{<\omega}$. Recursively on β we construct a sequence $\langle E_\beta \rangle_{\beta < \omega_1}$ such that

1. $n < \omega, \{\beta_1, \ldots \beta_n\} \subset \omega_1 \longrightarrow |E_{\beta_1} \cap \ldots \cap E_{\beta_n}| = \omega$,
2. $\beta < \omega_1 \longrightarrow \langle s_\beta, E_\beta \rangle \; ||^R \; A_{\alpha_\beta}^N$.

Assume that we have constructed $\langle E_\eta \rangle_{\eta < \beta}$. As this is countable and satisfies the first condition in the recursion hypothesis, there is $X \in [\omega]^\omega$ such that $\forall \eta < \beta \, (|X \setminus E_\eta| < \omega)$. Since $A_{\alpha_\beta}^N$ is completely Ramsey, there is $E_\beta \in [X \setminus \cup s_\beta]^\omega$ such that $\langle s_\beta, E_\beta \rangle \; ||^R \; A_{\alpha_\beta}^N$. Here we again need $N \models \mathrm{AC}$. That $\langle E_\eta \rangle_{\eta < \beta+1}$ satisfies the above conditions is easily checked. $\mathcal{F}$ is generated from the E_β's and cofinite subsets of ω by finite intersections. $\square$

Now we are on the point of defining our key forcing notion.

$$P_\mathcal{F} \stackrel{\text{def}}{=} \{\langle s, U \rangle \in P_M \mid U \in \mathcal{F}\}$$

ordered by $\leq_{P_M}$.

Claim 2.2 *$P_\mathcal{F}$ has the c.c.c.*

Proof of Claim: Let $\langle s_\beta, U_\beta \rangle_{\beta < \omega_1} \subseteq P_\mathcal{F}$. Since $|[\omega]^{<\omega}| = \omega$, we have β_0 and β_1 such that $s_{\beta_0} = s_{\beta_1}$. Then by condition 2 in Claim 2.1, $\langle s_{\beta_0}, U_{\beta_0} \cap U_{\beta_1} \rangle \in P_\mathcal{F}$ and is a common extension of $\langle s_{\beta_0}, U_{\beta_0} \rangle$ and $\langle s_{\beta_1}, U_{\beta_1} \rangle$. $\square$

Define

$$\begin{aligned} \text{for } \alpha < \omega_1, \; D_\alpha &\stackrel{\text{def}}{=} \{\langle s, U \rangle \in P_\mathcal{F} \mid \forall t \subseteq s \, (\langle t, U \rangle \; ||^R \; A_\alpha^N)\} \, , \\ \text{for } n < \omega, \; \overline{D_n} &\stackrel{\text{def}}{=} \{\langle s, U \rangle \in P_\mathcal{F} \mid s \not\subseteq n\} \, . \end{aligned}$$

Claim 2.3 *Each D_α is dense.*

Proof of Claim: Fix $\alpha < \omega_1$ and $\langle s, U \rangle \in P_\mathcal{F}$. Enumerate $\mathcal{P}(s)$ as $\{t_i\}_{i<n}$ with $n < \omega$. By condition 4 in Claim 2.1, for each $i < n$, there is $U_i \in \mathcal{F}$ such that $\langle t_i, U_i \rangle \; ||^R \; A_\alpha^N$. Using condition 2 in Claim 2.1 n-times, $\tilde{U} \stackrel{\text{def}}{=} U \cap \bigcap_{i<n} U_i$ is in $\mathcal{F}$. Then $\langle s, U \rangle \geq \langle s, \tilde{U} \rangle \in D_\alpha$. $\square$

Claim 2.4 *Each* $\overline{D_n}$ *is dense.*

Proof of Claim: Fix $n < \omega$ and $\langle s, U\rangle \in P_{\mathcal{F}}$. Let m be the least member of $U \setminus n$. Then $\langle s, U\rangle \geq \langle s \cup \{m\}, U \setminus (m+1)\rangle \in \overline{D_n}$. $\square$

Now we step out from N to the universe V. Since $|N| \leq \omega$, we have in V a $P_{\mathcal{F}}$-generic G over N. We can make $N[G]$ satisfy arbitrarily large finite fragments of ZFC by means of making the grand model N satisfy some large fragment. And since $N \models P_{\mathcal{F}}$ has the c.c.c., $\omega_1^N = \omega_1^{N[G]}$.

We argue in $N[G]$ from here. We define from φ, θ_φ and a, $A^{N[G]}$ and $A_\alpha^{N[G]}$ for $\alpha < \omega_1^{N[G]}$ in $N[G]$. Again, we can assume $N[G]$ knows (Lemma 5)

$$A^{N[G]} = \bigcup_{\alpha < \omega_1^{N[G]}} A_\alpha^{N[G]} .$$

Let X be $\bigcup\{s \mid \langle s, U\rangle \in G\}$. It is easily checked that $X \in [\omega]^\omega$ follows from $G \cap \overline{D_n} \neq 0$ for all n. And for all $\langle s, U\rangle \in G$, $X \in [s, U]$.

Claim 2.5 $\forall \alpha < \omega_1^{N[G]}$ ($A_\alpha^{N[G]} \cap [X]^\omega$ *is open in* $[X]^\omega$).

Proof of Claim: Fix $\alpha < \omega_1^{N[G]} = \omega_1^N$, and a real $c_\alpha \in N$ coding α. As D_α is dense, there is $\langle s, U\rangle \in G \cap D_\alpha$. Then $X \in [s, U]$. By the definition of D_α, for each $t \subseteq s$,

$$N \models \langle t, U\rangle \mid\mid^R A_\alpha^N .$$

(Here note that this is A_α^N, not $A_\alpha^{N[G]}$ yet.) But by the definition of A_α^N, this is written more verbosely by,

$$\begin{aligned} & N \models \forall z \in [t, U]\ \theta_\varphi(z, a, c_\alpha) \\ \vee\ \ & N \models \forall z \in [t, U]\ \chi_\varphi(z, a, c_\alpha) . \end{aligned}$$

Here "$z \in [t, U]$" is arithmetical, θ_φ and χ_φ are Π_1^1. So the whole statements "$\forall z \in [t, U] \ldots$" are both Π_1^1. Since N and $N[G]$ are models of a sufficiently large fragment of ZF, we may assume that these relations are absolute for N and $N[G]$ (Lemma 3). Hence

$$\begin{aligned} & N[G] \models \forall z \in [t, U]\ \theta_\varphi(z, a, c_\alpha) \\ \vee\ \ & N[G] \models \forall z \in [t, U]\ \chi_\varphi(z, a, c_\alpha) , \end{aligned}$$

that is,

$$\langle t, U\rangle \mid\mid^R A_\alpha^{N[G]} .$$

For any $Y \in [X]^\omega$, $Y \in [Y \cap s, X \setminus s] \subseteq [Y \cap s, U]$, so $Y \in A_\alpha^{N[G]}$ *iff* $\langle Y \cap s, U\rangle \mid\vdash^R A_\alpha^{N[G]}$. This implies

$$A_\alpha^{N[G]} \cap [X]^\omega = \bigcup\Big\{[t, X \setminus s] \mid t \subseteq s \wedge \langle t, U\rangle \mid\vdash^R A_\alpha^{N[G]}\Big\} ,$$

and this is a (finite) union of open sets in $[X]^\omega$. $\square$

Hence $A^{N[G]} \cap [X]^\omega = \bigcup_{\alpha<\omega_1}(A_\alpha^{N[G]} \cap [X]^\omega)$ is open in $[X]^\omega$. Moreover as the space ${}^\omega 2$ satisfies the second countability axiom (in fact its basic open sets are countable), so does the subspace $[X]^\omega$. Thus there exists $\gamma < \omega_1^{N[G]}$ such that

$$A_\gamma^{N[G]} \cap [X]^\omega = \bigcup_{\alpha<\gamma}(A_\alpha^{N[G]} \cap [X]^\omega) = \bigcup_{\alpha<\omega_1}(A_\alpha^{N[G]} \cap [X]^\omega) = A^{N[G]} \cap [X]^\omega \ .$$

We fix this γ.

Claim 2.6 *There is $Y \in [X]^\omega$ such that $\langle 0, Y\rangle \ ||^R \ A^{N[G]}$.*

Proof of Claim: As $A^{N[G]} \cap [X]^\omega$ is open in $[X]^\omega$, there is an open set $O \subseteq {}^\omega 2$ such that $O \cap [X]^\omega = A^{N[G]} \cap [X]^\omega$. We can assume that $N[G]$ knows that the open set O is completely Ramsey (Theorem 1). Now we have $Y \in [X]^\omega$ such that $\langle 0, Y\rangle \ ||^R \ O$, but this implies $\langle 0, Y\rangle \ ||^R \ A^{N[G]}$. $\square$

Here we already know that $N[G]$ thinks $A^{N[G]}$ is Ramsey. We shall show this situation also holds in V by absoluteness.

Now we step out from $N[G]$ to V. And we consider two cases.

Case 1: $N[G] \models \langle 0, Y\rangle \ |\!\vdash^R ({}^\omega 2 \setminus A^{N[G]})$.

This means

$$N[G] \models \forall z \in [Y]^\omega \ \neg\varphi(z, a) \ ,$$

but this is a Π_1^1 statement on Y and a. We can assume that Π_1^1 statements are absolute for $N[G]$ and V, so

$$V \models \forall z \in [Y]^\omega \ \neg\varphi(z, a) \ .$$

Thus $\langle 0, Y\rangle \ |\!\vdash^R ({}^\omega 2 \setminus A^V)$.

Case 2: $N[G] \models \langle 0, Y\rangle \ |\!\vdash^R A^{N[G]}$.

If we were to follow the same transformation as Case 1, we would have a Π_2^1 statement, which is not necessarily absolute for our countable model $N[G]$. So we use the above γ here. In fact,

$$N[G] \models \langle 0, Y\rangle \ |\!\vdash^R A_\gamma^{N[G]} \ .$$

Using $c_\gamma \in N[G]$ coding γ, this is written as

$$N[G] \models \forall z \in [Y]^\omega \ \theta_\varphi(z, a, c_\gamma) \ .$$

This is a Π_1^1 statement about Y, a and c_γ. This can be assumed to be absolute for $N[G]$ and V, thus

$$V \models \forall z \in [Y]^\omega \ \theta_\varphi(z, a, c_\gamma) \ .$$

This means $\langle 0, Y\rangle \mid\!\vdash^R A^V_\gamma$, and this clearly implies

$$\langle 0, Y\rangle \mid\!\vdash^R A^V \ .$$

In each case, $\langle 0, Y\rangle$ forces the same in $N[G]$ and V, and the theorem is proved. $\square$

The main difference between Silver's proof [2] and ours is as follows: Silver brought a forcing which makes $N[G] \models \mathrm{MA}(\omega_1)$, where $\mathrm{MA}(\omega_1)$ means that for every c.c.c. p.o. set and for every sequence of dense subsets whose length is at most ω_1, there exists a "generic" filter which intersects all dense sets in the sequence. And in $N[G]$ he constructed our $P_{\mathcal{F}}$ and directly got a "generic" object using $\mathrm{MA}(\omega_1)$. Since the cardinality of the family of dense subsets we want to make this "generic" subset intersect is ω_1, $\mathrm{MA}(\omega_1)$ suffices. His proof is mainly in $N[G]$, so does not need any argument about absoluteness for N and $N[G]$.

On the other hand, we can use Mathias forcing P_M instead of $P_{\mathcal{F}}$. Then we owe the preservation of ω_1 to Mathias [4, 4.10].

But the simplicity of our forcing will allow us to get some refinement in the case of $\mathbf{\Sigma}^1_2$.

4 On provably $\mathbf{\Delta}^1_2$ sets

In this section, we get the Ramseyness of a little wider class than the previous section. We first define the class of provably $\mathbf{\Delta}^1_2$ sets.

Definition 1 For $a \in {}^\omega 2$ and $A \subseteq {}^\omega 2$, A is (ZF-)*provably* $\Delta^1_2(a)$ *iff* there exist a Σ^1_2 formula $\varphi(x, w)$ and a Π^1_2 formula $\psi(x, w)$ such that:

- $x \in A$ *iff* $\varphi(x, a)$ *iff* $\psi(x, a)$,
- $\mathrm{ZF} \vdash \forall^1 x, w\, [\varphi(x, w) \leftrightarrow \psi(x, w)]$.

A is *provably* $\mathbf{\Delta}^1_2$ *iff* for some $a \in {}^\omega 2$, A is provably $\Delta^1_2(a)$.

Provably $\mathbf{\Delta}^1_2$ sets are also $\mathbf{\Delta}^1_2$, but the converse is false. The next lemma tells us that the theory used for the equivalence may be somewhat ambiguous.

Lemma 7 *A is* ZFC*-provably* $\Delta^1_2(a)$ *iff* *A is* ZF*-provably* $\Delta^1_2(a)$*. In fact, for any* Σ^1_2 φ *and* Π^1_2 ψ*,*

$$\begin{aligned} & \mathrm{ZFC} \vdash \forall^1 x, w\, [\varphi(x, w) \leftrightarrow \psi(x, w)] \\ \textit{iff} \quad & \mathrm{ZF} \vdash \forall^1 x, w\, [\varphi(x, w) \leftrightarrow \psi(x, w)] \ . \end{aligned}$$

Proof: Suppose ZFC $\vdash \forall^1 x, w\,[\varphi(x,w) \leftrightarrow \psi(x,w)]$, φ is Σ^1_2 and ψ is Π^1_2. It is enough to show that ZF implies the same theorem. "$\forall^1 x, w\,(L[x,w] \models \sigma_2)$" and "$\forall^1 x, w\,(\omega_1^V \subseteq L[x,w])$" are theorems of ZF, where σ_2 is as in Lemma 4. So by this theorem—in fact by one theorem in the schema,

$$\mathrm{ZF} \vdash \forall^1 x, w\ (\varphi(x,w) \leftrightarrow L[x,w] \models \varphi(x,w))\ .$$

Similarly,

$$\mathrm{ZF} \vdash \forall^1 x, w\ (\psi(x,w) \leftrightarrow L[x,w] \models \psi(x,w))\ .$$

And since "$L[x,w]$ satisfies a fragment of ZFC sufficient for a proof of $\forall^1 x, w\,[\varphi(x,w) \leftrightarrow \psi(x,w)]$" is a theorem of ZF, thus

$$\mathrm{ZF} \vdash \forall^1 x, w\ (L[x,w] \models \varphi(x,w) \leftrightarrow L[x,w] \models \psi(x,w))\ .$$

These equivalences establish our goal. $\square$

Note that "ZFC" is replaced by "ZF+$\exists^1 x(V = L[x])$", but not by "ZF+$V = L$".

Theorem 3 *Every provably $\mathbf{\Delta}^1_2$ sets of ${}^\omega 2$ is completely Ramsey.*

Proof: The outline of the proof is the same as the case of $\mathbf{\Sigma}^1_1$. We need to treat each absoluteness carefully because hierarchies are raised now.

We have only to show Ramseyness again. For if A is provably $\mathbf{\Delta}^1_2$ and $f : {}^\omega 2 \to {}^\omega 2$ is continuous then $f^{-1}(A)$ is provably $\mathbf{\Delta}^1_2$.

Now we fix $a \in {}^\omega 2$, and a provably $\Delta^1_2(a)$ set

$$\{x \mid \varphi(x,a)\} = A^V = {}^\omega 2 \setminus B^V = \{x \mid \neg\psi(x,a)\}\ ,$$

where φ and ψ are Σ^1_2. These are canonically divided into $\aleph_1$ Borel sets.

$$\begin{aligned} A^V &= \bigcup_{\alpha<\omega_1^V} A^V_\alpha \\ B^V &= \bigcup_{\alpha<\omega_1^V} B^V_\alpha\ , \end{aligned}$$

where A^V_α, B^V_α are defined from $\theta_\varphi, \theta_\psi$ and a. We want to show this A^V is Ramsey.

Again we begin our argument in N which is a countable transitive model of a fragment of ZF+AC sufficiently large for this proof with $a \in N$. We define (in N) $A^N, B^N, A^N_\alpha, B^N_\alpha$ for $\alpha < \omega_1^N$ using the same $\varphi, \psi, \theta_\varphi, \theta_\psi$ and a. N can be assumed to know that the splitting of Σ^1_2's (Lemma 6) and that A is provably $\Delta^1_2(a)$:

$$A^N = \bigcup_{\alpha<\omega_1^N} A^N_\alpha = {}^\omega 2 \setminus B^N = {}^\omega 2 \setminus \bigcup_{\alpha<\omega_1^N} B^N_\alpha\ .$$

Moreover N can be assumed to think each A_α^N, B_α^N is Borel and so completely Ramsey (Theorem 1).

We construct $\mathcal{F}$, $P_\mathcal{F}$, D_α and $\overline{D_n}$ similarly to the previous section but with another property of $\mathcal{F}$:

5. $\forall\alpha < \omega_1 \forall s \in [\omega]^{<\omega} \exists U \in \mathcal{F}\, [\langle s, U\rangle \,||^R\, B_\alpha^N]$.

The construction is straightforward. We can check that $P_\mathcal{F}$ has the c.c.c. and that D_α and $\overline{D_n}$ are dense.

Next we step out to V and get a $P_\mathcal{F}$-generic G over N. From now on we argue in $N[G]$, which is a model of an arbitrarily large fragment of ZF+AC. Since $N \models P_\mathcal{F}$ has the c.c.c., $\omega_1^N = \omega_1^{N[G]}$.

Now in $N[G]$, we define $A^{N[G]}, B^{N[G]}, A_\alpha^{N[G]}$, and $B_\alpha^{N[G]}$ for $\alpha < \omega_1^{N[G]}$ by the same formulae $\varphi, \psi, \theta_\varphi, \theta_\psi$ and a. Here again $N[G]$ can be assumed to know

$$A^{N[G]} = \bigcup_{\alpha<\omega_1^{N[G]}} A_\alpha^{N[G]} = {}^\omega 2 \setminus B^{N[G]} = {}^\omega 2 \setminus \bigcup_{\alpha<\omega_1^{N[G]}} B_\alpha^{N[G]} \,.$$

Let X be $\bigcup\{s \mid \langle s, U\rangle \in G\}$. Then $X \in [\omega]^\omega$ by the density of $\overline{D_n}$.

The following claim is shown by the same proof as before.

Claim 3.1 $\forall\alpha < \omega_1$ *(*$A_\alpha^{N[G]} \cap [X]^\omega$ *and* $B_\alpha^{N[G]} \cap [X]^\omega$ *are open in* $[X]^\omega$*).*

Again, $A^{N[G]} \cap [X]^\omega = \bigcup_{\alpha<\omega_1^{N[G]}}(A_\alpha^{N[G]} \cap [X]^\omega)$ and $B^{N[G]} \cap [X]^\omega = \bigcup_{\alpha<\omega_1^{N[G]}}(B_\alpha^{N[G]} \cap [X]^\omega)$ are open (in fact clopen) in $[X]^\omega$, and by the second countability axiom there exists $\gamma < \omega_1^{N[G]}$ such that

$$A^{N[G]} \cap [X]^\omega = A_\gamma^{N[G]} \cap [X]^\omega = [X]^\omega \setminus B^{N[G]} = [X]^\omega \setminus B_\gamma^{N[G]} \,.$$

We fix this γ and a real c_γ coding γ. Because $A^{N[G]}$ is open in $[X]^\omega$ and $N[G]$ thinks open sets are completely Ramsey (Theorem 1), there is $Y \in [X]^\omega$ such that $\langle 0, Y\rangle \,||^R\, A^{N[G]}$.

Now we step out from $N[G]$ to V. And we consider two cases.

Case 1: $N[G] \models \langle 0, Y\rangle \,||\!\!-^R A^{N[G]}$.

In fact,

$$N[G] \models \langle 0, Y\rangle \,||\!\!-^R A_\gamma^{N[G]} \,,$$

that is,

$$N[G] \models \forall z \in [Y]^\omega\, \theta_\varphi(z, a, c_\gamma) \,.$$

This statement is Π^1_1 as before, so this is assumed to be absolute for $N[G]$ and V, thus

$$V \models \forall z \in [Y]^\omega\, \theta_\varphi(z, a, c_\gamma) \,.$$

This means $\langle 0, Y\rangle \mid\!\vdash^R A^V_\gamma$ and implies

$$\langle 0, Y\rangle \mid\!\vdash^R A^V .$$

Case 2: $N[G] \models \langle 0, Y\rangle \mid\!\vdash^R ({}^\omega 2 \setminus A^{N[G]})$.

This case is similar to the previous case, using B_γ.

The rest is the same as the previous section. □

5 On $\mathbf{\Sigma}^1_2$ sets

We go on to $\mathbf{\Sigma}^1_2$ sets. Unlike before, we cannot prove the Ramseyness of $\mathbf{\Sigma}^1_2$ sets from ZF alone because if ${}^\omega 2 \subseteq L$ then there exists a Δ^1_2 set which is not Ramsey. Hence by our second result (Theorem 3), this set is also a counterexample of Δ^1_2 and not provably $\mathbf{\Delta}^1_2$. So we need some additional assumption which implies ${}^\omega 2 \not\subseteq L$ at least.

Silver in [2] started from the existence of a measurable cardinal, and essentially used the assumption that for each $a \subseteq \omega$, ω_1^V is inaccessible in $L[a]$. He showed this assumption implies that all $\mathbf{\Sigma}^1_2$ sets are Ramsey.

We use our forcing and make this theorem effective and somewhat refined.

Theorem 4 *For $a \in {}^\omega 2$, if $\omega_1^{L[a]} < \omega_1$ then every $\Sigma^1_2(a)$ subset of ${}^\omega 2$ is Ramsey.*

Proof: Because of the effectiveness of our statement, we cannot establish complete Ramseyness. (If A is $\Sigma^1_2(a)$ and f is a continuous mapping coded by a real z_f, $f^{-1}(A)$ would be $\Sigma^1_2(a, z_f)$.)

Given a $\Sigma^1_2(a)$ set $A^V = \{x \mid \varphi(x, a)\}$ with a Σ^1_2 formula φ, using Lemma 6, we define

$$A^V_\alpha = \{x \mid \text{ for some (any) } z \text{ coding } \alpha,\ \theta_\varphi(x, a, z)\} .$$

We know $A^V = \bigcup_{\alpha<\omega_1} A^V_\alpha$. We shall show that this A^V is Ramsey.

We start from $L[a]$, which is the class of constructible sets from a, as a ground model. In $L[a]$ the whole ZFC holds. From now on we argue in $L[a]$. $L[a]$ knows $\mathbf{\Sigma}^1_2$ can be split into $\aleph_1$ Borel sets (Lemma 6). We can define $A^{L[a]}$ and $A^{L[a]}_\alpha$'s by the same formulae φ, θ_φ, and a as in V. We know

$$A^{L[a]} = \bigcup_{\alpha<\omega_1^{L[a]}} A^{L[a]}_\alpha .$$

$L[a]$ knows that each $A^{L[a]}_\alpha$ is Borel and hence is completely Ramsey (by Theorem 1). This implies the same Claim as Claim 2.1, and we have $\mathcal{F}$, c.c.c. $P_\mathcal{F}$, and dense subsets D_α, $\overline{D_n}$.

Now again we step out from $L[a]$ to the universe V. Can we find a $P_{\mathcal{F}}$-generic G over $L[a]$? G must touch all dense subsets of $P_{\mathcal{F}}$ in $L[a]$. But as $L[a]$ satisfies AC, we have only to make G touch all maximal antichains of $P_{\mathcal{F}}$ in $L[a]$. Moreover as $L[a] \models P_{\mathcal{F}}$ has the c.c.c., the cardinality of each maximal antichain is at most countable in $L[a]$. And $L[a]$ knows $|P_{\mathcal{F}}| \leq \omega_1^{L[G]}$ and the continuum hypothesis $2^\omega = \omega_1$. Consequently,

$$\left|\{D \in L[a] \mid D \text{ is maximal antichain in } P_{\mathcal{F}}\}\right| \leq \left|({}^{\omega}P_{\mathcal{F}})^{L[a]}\right| \leq ({\omega_1}^{\omega})^{L[a]} \leq (\omega_1)^{L[a]} < \omega_1^V \ .$$

This tells us that we have a $P_{\mathcal{F}}$-generic G over $L[a]$ in V. Again since $L[a] \models P_{\mathcal{F}}$ has the c.c.c., $\omega_1^{L[a]} = \omega_1^{L[a][G]}$.

We argue in $L[a][G]$ from here. $L[a][G]$ knows that $A^{L[a][G]}$ is the union of $A_\alpha^{L[a][G]}$'s and that these are well-defined. Let X be $\bigcup\{s \mid \langle s, U\rangle \in G\}$. Note that for all $\langle s, U\rangle \in G$, $X \in [s, U]$. Since $\overline{D_n}$ is dense, $X \in [\omega]^\omega$.

Claim 4.1 $\forall \alpha < \omega_1$ *(*$A_\alpha^{L[a][G]} \cap [X]^\omega$ *is open in* $[X]^\omega$ *).*

The proof of this claim is exactly as before. Hence $A^{L[a][G]} \cap [X]^\omega = \bigcup_{\alpha < \omega_1^{L[a][G]}}(A_\alpha^{L[a][G]} \cap [X]^\omega)$ is open in $[X]^\omega$. And again by the second countability axiom there exists $\gamma < \omega_1^{L[a][G]}$ such that

$$A_\gamma^{L[a][G]} \cap [X]^\omega = \bigcup_{\alpha<\gamma}(A_\alpha^{L[a][G]} \cap [X]^\omega) = \bigcup_{\alpha<\omega_1^{L[a][G]}}(A_\alpha^{L[a][G]} \cap [X]^\omega) = A^{L[a][G]} \cap [X]^\omega \ .$$

We fix this γ and a real c_γ coding this.

$A^{L[a][G]}$ is open in $[X]^\omega$ implies that there is $Y \in [X]^\omega$ such that $\langle 0, Y\rangle \ ||^R \ A^{L[a][G]}$.

Now we step out from $L[a][G]$ to V. We consider two cases here, too.

Case 1: $L[a][G] \models \langle 0, Y\rangle \ ||\!\!-^R \ ({}^{\omega}2 \setminus A^{L[a][G]})$

This means

$$L[a][G] \models \forall z \in [Y]^\omega \ \neg\varphi(z, a) \ ,$$

but this is a Π^1_2 statement about Y and a and by Shoenfield's $\mathbf{\Sigma}^1_2$ absoluteness theorem (Lemma 4) this statement is absolute for $L[a][G]$ and V, so

$$V \models \forall z \in [Y]^\omega \ \neg\varphi(z, a) \ .$$

Thus $\langle 0, Y\rangle \ ||\!\!-^R \ ({}^{\omega}2 \setminus A^V)$.

Case 2: $L[a][G] \models \langle 0, Y\rangle \ ||\!\!-^R \ A^{L[a][G]}$

Using the above γ,

$$L[a][G] \models \langle 0, Y\rangle \ ||\!\!-^R \ A_\gamma^{L[a][G]} \ .$$

This means

$$L[a][G] \models \forall z \in [Y]^\omega\ \theta_\varphi(z, a, c_\gamma)\ .$$

This is a Π^1_1 statement about Y, a and c_γ, so this is absolute for $L[a][G]$ and V,

$$V \models \forall z \in [Y]^\omega\ \theta_\varphi(z, a, c_\gamma)\ ,$$

and this implies $\langle 0, Y\rangle \mid\mid\!\!-^R A^V$.

In each case, $\langle 0, Y\rangle \mid\mid^R A^V$, and the theorem is proved. $\square$

Remark: If we use a forcing making $\mathrm{MA}(\omega_1)$ as Silver did, then this forcing notion has the c.c.c. and is of cardinality ω_2. Thus in $L[a]$ the number of all maximal antichains is ω_2. We would need the assumption that $\omega_2^{L[a]} < \omega_1$.

If we used Mathias forcing P_M, then since P_M does not satisfy the c.c.c., we would need $\omega_2^{L[a]} < \omega_1$ again.

We also have a corollary which is related to a large cardinal hypothesis.

Corollary 5 (Silver) *If $\forall a\, (\omega_1^{L[a]} < \omega_1)$ then every $\mathbf{\Sigma}^1_2$ set of ${}^\omega 2$ is completely Ramsey.*

Proof of Corollary: Suppose A is $\Sigma^1_2(a)$ and $f : {}^\omega 2 \to {}^\omega 2$ is continuous. f can be coded by one real, say z_f. Then $f^{-1}(A)$ is $\Sigma^1_2(a, z_f)$. So our theorem tells us $f^{-1}(A)$ is Ramsey. This shows that A is completely Ramsey. $\square$

The converse of our third theorem (Theorem 4) is false: From Silver's lemma ([2, p. 61 LEMMA]) we know

$$\mathrm{MA}(\omega_1) \longrightarrow \text{“all } \mathbf{\Sigma}^1_2 \text{ sets are Ramsey”}\ ,$$

so the right statement is equiconsistent with ZF. But it is easily checked that

$$\forall^1 a\, (\omega_1^{L[a]} < \omega_1^V) \longrightarrow \forall^1 a\, (\omega_1^V \text{ is inaccessible in } L[a])\ ,$$

which is consistency-wise stronger than ZF.

This result is optimal in the sense of height in the projective hierarchy: If $V = L[\mu]$ holds, in other words if the universe is the canonical inner model for measurability, then there is a Δ^1_3 well-ordering of ${}^\omega 2$, so there is a Δ^1_3 non-Ramsey set ([5, 20.18, 20.20]). But this hypothesis implies that $\forall^1 a\, (\omega_1^{L[a]} < \omega_1)$.

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